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LINEAR PROGRAMMING AND PRIMAL - DUAL PROBLEMS IN GRAPH THEORY

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Abstract. One of the most important and applied concepts in graph theory is to find the edge cover, vertex cover and dominating sets with minimum cardinal also to find independence and matching sets with maximum cardinal.
In this paper we will study all of these concepts from viewpoint linear programming and *primal* - *dual* problem.

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1. Introduction

All graphs in this note are simple, connected, finite and undirected. Let $G = (V, E)$ be a simple and connected graph with $|V| = n$ and $|E| = m$.
For some notation is not defined here we refer the reader to [6].
A set $L \subseteq E$ is an *edge cover* if every vertex $v \in V$ is incident to some edge of L.
A set $Q \subseteq V$ is a *vertex cover* if every edge $e \in E$ has at least one endpoint in Q.
A set $S \subseteq V$ is an *independent set* if non of two vertex in s are not adjacent.
The maximum size of an independent set is named *independence number*.
A *matching* in graph G is a set $M \subseteq E$ with no shared endpoints.
We set

$$Min|L| = \beta',$$

$$Max|S| = \alpha,$$

$$Min|Q| = \beta,$$

$$Max|M| = \alpha'.$$

One can show that [6]

$$(i) \quad \alpha + \beta = n,$$

$$(ii) \quad \alpha' + \beta' = n.$$

we denote the *adjacency matrix* with A and *incidence matrix* with R, in which $A = [a_{ij}]_{n\times n}$;

$$a_{ij} = the\ numbers\ of\ edges\ with\ endpoints\ v_i, v_j,$$

and $R = [r_{ij}]_{n\times m}$;

$$r_{ij} = \begin{cases} 1 & v_i\ is\ an\ endpoint\ of\ e_j, \\ 0 & otherwise. \end{cases}$$

From now on we set

$$V = (v_1, v_2, ..., v_n)^t,$$

$$E = (e_1, e_2, ..., e_m)^t,$$

and

$$1_n = (1, 1, ..., 1)^t_{1\times n}.$$

The four following theorems proved by authors in[1].
As above notation we have the following theorems for obtaining the $\beta^{'}$, α, β, $\alpha^{'}$ respectively.

Theorem 1.1

$$\beta^{'}(G) = \min \sum_{i=1}^{m} e_i$$

$$\textit{subject to:} \qquad RE \geq 1_n, \qquad\qquad P(1)$$

$$e_i \in \{0, 1\} \quad i = 1, 2, ..., m.$$

Theorem 1.2

$$\alpha(G) = \max \sum_{i=1}^{n} v_i$$

$$\textit{subject to:} \qquad R^tV \leq 1_m, \qquad\qquad P(2)$$

$$v_i \in \{0, 1\} \quad i = 1, 2, ..., n.$$

Theorem 1.3

$$\beta(G) = \min \sum_{i=1}^{n} v_i$$
$$\textit{subject to:} \quad R^t V \geq 1_m, \qquad P(3)$$
$$v_i \in \{0,1\} \quad i = 1,2,...,n.$$

Theorem 1.4

$$\alpha^{'}(G) = \max \sum_{i=1}^{m} e_i$$
$$\textit{subject to:} \quad RE \leq 1_n, \qquad P(4)$$
$$e_j \in \{0,1\} \quad j = 1,2,...,m.$$

2.MAIN RESULT

We consider the problems $P(1)$ and $P(2)$ and delete the conditions $e_j \in \{0,1\}$, $j = 1,2,...,m$ and $v_i \in \{0,1\}$ $i = 1,2,...,n$ temporarily and replacement the conditions $e_j \geq 0$ and $v_i \geq 0$ respectively. So $P(1)$, $P(2)$ are *primal - dual* problems in linear programming , then we have the two following theorems for them.

Theorem 2.1 (*Weak duality property*) *The objective function value for any feasible solution to the minimization problem* ($\beta' = \sum_{i=1}^{m} e_i$) *is always greater than or equal to the objective function value for any feasible solution to maximization problem* ($\alpha = \sum_{i=1}^{n} v_i$) *.*

Proof. See [2].

Theorem 2.2 *If there are feasible solutions for primal and dual problems such that their objective functions are the same then this solutions are the optimum for primal and dual problem.*

Proof. See [2].

Now if $e_j \in \{0,1\}$ and $v_i \in \{0,1\}$ satisfy in $P(1)$ *and* $P(2)$ respectively, then we call them feasible solutions for $p(1)$ and $P(2)$. Hence by Theorems 2.1 and 2.2 for each graph G we have:

(i) $\beta' \geq \alpha,$ (1)
(ii) if $\sum_{i=1}^{m} e_i = \sum_{i=1}^{n} v_i = \lambda$ *then* $\beta' = \alpha = \lambda.$
On the other hand since in every column of matrix R the entry 1 appears twice exactly, (*i.e.* every edge has two unique endpoints), the system $RE \geq 1_n$ implies that:

$$2(e_1 + e_2 + \cdots + e_m) \geq n,$$

or

$$(e_1 + e_2 + \cdots + e_m) \geq n/2,$$

and we obtain another lower bound for β' as follow:

$$\beta' \geq \lceil \frac{n}{2} \rceil. \qquad (2)$$

With the same argument in above about the problems $P(3)$ and $P(4)$ we have:
(iii) $\beta \geq \alpha',$ (3)
(iv) if $\sum_{i=1}^{m} e_i = \sum_{i=1}^{n} v_i = \eta$ *then* $\beta = \alpha' = \eta.$
Also from $RE \leq 1_n$ implies that :

$$2(e_1 + e_2 + \cdots + e_m) \leq n,$$

or

$$(e_1 + e_2 + \cdots + e_m) \leq n/2,$$

and we obtain another upper bound for α' as follow:

$$\alpha' \leq \lfloor \frac{n}{2} \rfloor. \qquad (4)$$

From (1) to (4) for every graph G we have two inequalities as follow:

$$\beta + \beta' \geq n,$$
$$\alpha + \alpha' \leq n.$$

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