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UNIFORMLY STARLIKE AND CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. Let $A(\omega)$ be the class of analytic functions of the form:

$$f(z) = (z-\omega) + \sum_{k=2}^{\infty} a_k (z-\omega)^k$$

defined on the open unit disk $U = \{z : |z| < 1\}$ normalized with $f(\omega) = 0, f'(\omega) - 1 = 0$ and ω is an arbitrary fixed point in U. In this paper, we define a subclass of $\omega - \alpha - uniform$ starlike and convex functions by using a more generalized form of Ruschewey derivative operator. Several properties such as coefficient inequalities, extremal and distortion theorem, radii of starlike, convexity and close-to-convexity, convolution and integral operator are considered.

Keywords: Analytic, univalent, coefficient inequalities, extremal, distortion, convolution, integral operator, Ruscheweyh derivative, radii.

1. Introduction

Let $A(\omega)$ be the class of analytic functions of the form(see [7]):

$$f(z) = (z-\omega) + \sum_{k=2}^{\infty} a_k (z-\omega)^k \tag{1}$$

defined in the open unit disk $U = \{z : |z| < 1\}$, normalized with $f(\omega) = 0$ and $f'(\omega) - 1 = 0$. Let $S(\omega)$ denotes the subclass of $A(\omega)$ consisting of functions that are univalent in U and ω is an arbitrary fixed point in U. Let $S^*(\omega, \beta)$ and $S^c(\omega, \beta)$ be the classes of functions respectively ω-starlike of order β and ω-convex of order $\beta, (0 \leq \beta < 1)$ (see [10]). It is obvious to see that for $\omega = 0$ we have the usual class of all analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. A natural "Alexander relation" is preserved between the classes $S^*(\omega, \beta)$ and $S^c(\omega, \beta)$ and they are closely related to the ones defined by A.W. Goodman [5,6] except that, in our own case ω is an arbitrary fixed point in U.

Let $T(\omega)$ be the subclass of $S(\omega)$, consisting of functions of the form:

$$f(z) = (z-\omega) - \sum_{k=2}^{\infty} |a_k|(z-\omega)^k. \tag{2}$$

This function $T(\omega)$ is called a function with negative coefficient with ω arbitrarily fixed in U. In this paper we wish to study the following classes of functions:

Definition 1.1 *Let ω be an arbitrary fixed point in U, let $b \neq 0$, $m \in N_0, \alpha \geq 0, 0 \leq \beta < 1, l \geq 0, \lambda \geq 0$. We shall let $M_\omega^m(\alpha, \beta, \lambda, l, b)$ consist of functions $f \in T(\omega)$ satisfying the condition*

$$\mathrm{Re}\left[\frac{1}{b}\frac{(z-\omega)[I_\omega^m(\lambda,l)f(z)]'}{I_\omega^m(\lambda,l)f(z)}\right] > \alpha\left|\frac{1}{b}\frac{(z-\omega)[I_\omega^m(\lambda,l)f(z)]'}{I_\omega^m(\lambda,l)f(z)} - 1\right| + \beta \tag{3}$$

where $I_\omega^m(\lambda, l)$ denote the generalized Ruscheweyh derivatives which we are introducing here and is given as

$$I_\omega^m(\lambda,l)f(z) = \frac{(z-\omega)[(z-\omega)^{m-1}I_\omega(\lambda,l)]^{(m)}}{m!}, \ \ (m \in N_0).$$

Note that if f is given by (2), then we see that

$$I_\omega^m(\lambda,l)f(z) = (z-\omega) - \sum_{k=2}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]\Psi(m,k)|a_k|(z-\omega)^k, \tag{4}$$

where $\lambda \geq 0, l \geq 0, m \geq \in N_0$ and $\Psi(m,k) = \binom{k+m-1}{m}$.

This family of functions, that is $M_\omega^m(\alpha, \beta, \lambda, l, b)$ contains many well known as well as new classes of analytic univalent functions. With various special choices of parmeters involved, classes such as $M_0^m(\alpha, \beta, \lambda, 1) \equiv M_\lambda^n(\alpha, \beta)(m = n)$ class of functions studied by Al Shaqsi and Darus (see [3]), $M_0^1(\alpha, \beta, \lambda, 0, 1) \equiv U(k, \lambda, \beta)$ is the class of functions studied by Shanmugan et al (see [13]), $M_0^0(\alpha, 0, 0, 0, 1) = \alpha - ST, M_0^1(\alpha, 0, 1, 0, 1) \equiv \alpha - U \subset V$ respectively, the class of α-uniformly starlike function and α-uniformly convex function introduced and studied by Kannas and Wisniowska (see [8,9]), the classes $M_\omega^0(0, 0, 0, 0, 1) \equiv S^*(\omega)$ and $M_\omega^0(0, 0, 0, 0, 1) \equiv S^c(\omega)$ are respectively the classes of starlike and convex functions introduced and studied by Kannas and Ronning (see [7]), Acu and Owa (see [1]) and Aouf et al (see [4]). If ω is an arbitrary fixed point in U, and with various special choices of other parameters involved, many new clsses shall be obtained. As earlier said, in this paper we provide necessary and sufficient conditions such as coefficient bound, extreme point, radius of ω-close-to-convexity, ω-starlike and ω-convexity for functions in $M_\omega^m(\alpha, \beta, \lambda, l, b)$. Such inclusion involving Hadamard product (or convolution) and integral operator are also considered. Throughout this work we shall take $|\omega| = d$ (see [7], [2], [14]).

2. Coefficient inequality

Theorem 2.1 *Let f be given by (2) then, $f \in M_\omega^m(\alpha,\beta,\lambda,l,b)$ if and only if*

$$\sum_{k=2}^{\infty}(r+d)^{k-1}\Phi(k,\lambda,l)\Psi(m,k)[k-b\beta+\alpha(k-b)]|a_k| \le b(1-\beta). \tag{5}$$

where $\alpha,\lambda,l \ge 0,\ b>0$ (positive real), $m\in N_0,\ 0\le\beta<1,\ |\omega|=d$ and

$$\Phi(k,\lambda,l)=\frac{1+\lambda(k-1)+l}{1+l}\ \text{ and }\ \Psi(m,k)=\binom{k+m-1}{m}.$$

Proof: We would like to employ the technique adopted by Aglan et al (see [2]) to find the coefficient estimates for our class. We have $f\in M_\omega^m(\alpha,\beta,\lambda,l,b)$ if the condition (3) is satisfied. We claim the following fact

$$\text{Re}(\gamma)>\alpha|\gamma-1|+\beta \Leftrightarrow [\gamma(1+\alpha e^{i\theta})-\alpha e^{i\theta}]>\beta,\ \ -\pi\le\theta<\pi.$$

Based on the above, equation (3) may be written as

$$\text{Re}\left[\frac{1}{b}\frac{(z-\omega)[I_\omega^m(\lambda,l)f(z)]'}{I_\omega^m(\lambda,l)f(z)}(1+\alpha e^{i\theta})-\alpha e^{i\theta}\right]=$$

$$\text{Re}\left[\frac{(z-\omega)[I_\omega^m(\lambda,l)f(z)]'(1+\alpha e^{i\theta})-b\alpha e^{i\theta}I_\omega^m(\lambda,l)f(z)}{bI_\omega^m(\lambda,l)f(z)}\right]>\beta. \tag{6}$$

Next, we let

$$A(z)=(z-\omega)[I_\omega^m(\lambda,l)f(z)]'(1+\alpha e^{i\theta})-b\alpha e^{i\theta}I_\omega^m(\lambda,l)f(z),\ B(z)=bI_\omega^m(\lambda,l)f(z).$$

Then (6) is equivalent to $|A(z)+(1+\beta)B(z)|>|A(z)-(1+\beta)B(z)|$ for $0\le\beta<1$. For $A(z)$ and $B(z)$ as in above, we have

$$\Big|A(z)+(1-\beta)B(z)\Big| \ \ge\ \big[1+b(1-\beta)+\alpha(1-b)\big]\big|z-\omega\big|$$

$$-\sum_{k=2}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]\Phi(m,k)\big[k+b(1-\beta)+\alpha(k-b)\big]|a_k|\big|z-\omega\big|^k$$

and similarly,

$$\Big|A(z)-(1+\beta)B(z)\Big| \ \le\ \big[1-b(1-\beta)+\alpha(1-b)\big]\big|z-\omega\big|-$$

$$\sum_{k=2}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]\Phi(m,k)\big[k-b(1-\beta)+\alpha(k-b)\big]|a_k|\big|z-\omega\big|.$$

Therefore,

$$\big|A(z)+(1-\beta)B(z)\big|-\big|A(z)-(1+\beta)B(z)\big| \ \geq$$

$$2b(1-\beta)-2\sum_{k=2}^{\infty}(r+d)^{k-1}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]\Phi(m,k)\big[k-b\beta+\alpha(k-b)\big]|a_k|,$$

which yeild (5).

On the other hand,we must have

$$\mathrm{Re}\left[\frac{1}{b}\frac{(z-\omega)[I_{\omega}^{m}(\lambda,l)f(z)]'}{I_{\omega}^{m}(\lambda,l)f(z)}\big(1+\alpha e^{i\theta}\big)-\alpha e^{i\theta}\right] \ > \beta$$

On choosing the values of z on the positive real axis where $0\leq|z-\omega|=(r+d)<1$, the above inequality reduces to

$$\mathrm{Re}\left[\frac{\left[1-b\beta+\alpha e^{i\theta}(1-b)\right](r+d)-\sum\limits_{k=2}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]\Phi(m,k)\left[k-b\beta+\alpha e^{i\theta}(k-b)\right]|a_k|(r+d)^k}{b(r+d)-b\sum\limits_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)\Phi(m,k)|a_k|(r+d)^k}\right] \quad \geq \quad 0.$$

Since $\mathrm{Re}(-e^{i\theta})\geq-|e^{i\theta}|=-1$, the above inequality reduces to

$$\mathrm{Re}\left[\frac{(1-b\beta+\alpha(1-b))(r+d)-\sum_{k=2}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]\Phi(m,k)\left[k-b\beta+\alpha(k-b)\right]|a_k|(r+d)^k}{b(r+d)-\sum_{k=2}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\Phi(m,k)|a_k|(r+d)^k\right]}\right] \geq 0.$$

On setting $\Psi(k,\lambda,l)=\left(\frac{1+\lambda(k-1)+l}{1+l}\right)$, we have the required result, and the result is sharp with the extremal function f given by

$$f(z)=(z-\omega)-\frac{b(1-\beta)}{(r+d)^{k-1}\Phi(m,k)\Psi(k,\lambda,l)\big[k-b\beta+\alpha(k-b)\big]}(z-\omega)^k \tag{7}$$

Our next result is on the growth and distortion theorems.

Theorem 2.2 *Let the function f defined by (2) be in the class $M_{\omega}^{m}(\alpha,\beta,\lambda,l,b)$. Then for $|z-\omega|=r+d$, we have*

$$(r+d)-\frac{b(1-\beta)}{(m+1)\Psi(2,\lambda,l)\big[2-b\beta+\alpha(2-b)\big]}(r+d)\leq\big|f(z)\big| \ \leq$$

$$(r+d)+\frac{b(1-\beta)}{(m+1)\Psi(2,\lambda,l)\big[2-b\beta+\alpha(2-b)\big]}(r+d) \tag{8}$$

where $\Psi(2,\lambda,l) = \frac{1+\lambda+l}{1+l}$. *Equality holds for the function*

$$f(z) = (z-\omega) - \frac{b(1-\beta)}{(m+1)\Psi(2,\lambda,l)\big[2-b\beta+\alpha(2-b)\big](r+d)}(z-\omega)^2. \qquad (9)$$

Proof: Let us take the proof of the right hand side inequality in (8), because similar argument we would use for the first part is applicable to the other side.

$$\sum_{k=2}^{\infty}|a_k| \;\leq\; \frac{b(-\beta)}{(m+1)\Psi(2,\lambda,l)\big[2-b\beta+\alpha(2-b)\big]}$$

Since $f(z) = (z-\omega) - \sum\limits_{k=2}^{\infty}|a_k|(z-\omega)^k$

$$\big|f(z)\big| = \big|z-\omega\big| - \sum_{k=2}^{\infty}|a_k|\big|z-\omega\big|^k \;\leq\; (r+d) + \sum_{k=2}^{\infty}|a_k|(r+d)^k$$

$$\leq\; (r+d) + (r+d)^2\sum_{k=2}^{\infty}|a_k| \;\leq\; (r+d) + \frac{b(1-\beta)}{(m+1)\big[2-b\beta+\alpha(2-b)\big]\Psi(2,\lambda,l)}(r+d)$$

which yeilds the right hand side of (8).

By using the technique of Theorem 2.2, we give the distortion theorem.

Theorem 2.3 *Let the function* f *defined by (2) be in the class* $M_{\omega}^{m}(\alpha,\beta,\lambda,l,b)$. *Then for* $\big|z-\omega\big| = (r+d)$ *we have*

$$1 - \frac{b(1-\beta)}{(m+1)\Psi(2,\lambda,l)\big[2,-b\beta+\alpha(2-b)\big]} \;\leq\; \big|f'(z)\big|$$

$$\leq 1 + \frac{b(1-\beta)}{(m+1)\Psi(2,\lambda,l)\big[2-b\beta+\alpha(2-b)\big]}$$

Equality holds for function given by (9).

Theorem 2.4 $f \in M_{\omega}^{m}(\alpha,\beta,\lambda,l,b)$, *then* $f \in ST^{*}(\gamma)$, *where*

$$\gamma = 1 - \frac{b(k-1)(1-\beta)}{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)\big[k-b\beta+\alpha(k-b)-b(1-\beta)\big]}$$

Proof: It is sufficient to show that (5) implies

$$\sum_{k=2}^{\infty}(r+d)^{k-1}(k-\gamma)|a_k| \;\leq\; 1-\gamma$$

that is

$$\frac{k-\gamma}{1-\gamma} \ \leq \ \frac{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)\big[k-b\beta+\alpha(k-b)\big]}{b(1-\beta)}$$

then

$$\gamma \ \leq \ 1-\frac{b(k-1)(1-\beta)}{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)\big[k-b\beta+\alpha(k-b)\big]}$$

The above inequality holds true for $m \in N_0,\ k \geq 2\ \alpha,\beta,l \geq 0,\ b>0,\ 0 \leq \beta < 1$ and $|\omega| = d$. Also,

$$\Psi(k,\lambda,l) = \frac{1+\lambda(k-1)+l}{1+l},\ and\ \ \Phi(m,k) = \binom{k+m-1}{m}$$

Our next result is on the extreme points.

Theorem 2.5 *Let* $f_1(z) = (z-\omega)$ *and*

$$f_k(z) = (z-\omega) - \frac{b(1-\beta)}{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)\big[k-b\beta+\alpha(k-b)\big]}(z-\omega)^k,\ \ k<2$$

Then $f \in M_{\omega}^{m}(\alpha,\beta,\lambda,l,b)$, *if and only if it can be represented in the form*

$$f(z) = \sum_{k=1}^{\infty}\mu_k f_k(z), \qquad (\mu_k \geq 0,\ \sum_{k=1}^{\infty}\mu_k = 1). \tag{10}$$

Proof. We suppose $f(z)$ can be expressed as in (10). Then

$$f(z) = \sum_{k=1}^{\infty}\mu_k f_k(z) = \mu_1 f_1(z) + \sum_{k=2}^{\infty}\mu_k f_k(z) =$$

$$\mu_1 f_1(z) + \sum_{k=2}^{\infty}\left[(z-\omega) - \frac{b(1-\beta)}{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}\right] =$$

$$\mu_1(z-\omega) + \textstyle\sum_{k=2}^{\infty}\mu_k(z-\omega) - \sum_{k=2}^{\infty}\mu_k\left[\frac{b(1-\beta)}{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}(z-\omega)^k\right] \ =$$

$$(z-\omega) - \sum_{k=2}^{\infty}\mu_k\left[\frac{b(1-\beta)}{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}(z-\omega)^k\right]$$

Thus

$$= \sum_{k=2}^{\infty} \mu_k \left(\frac{b(1-\beta)}{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]} \right)$$

$$\times \left(\frac{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}{b(1-\beta)} \right)$$

$$= \sum_{k=2}^{\infty} \mu_k = \sum_{k=1}^{\infty} \mu_k - \mu_1 = 1 - \mu_1 \leq \ 1$$

So by Theorem 2.1, $f \in M_{\omega}^{m}(\alpha, \beta, \lambda, l, b)$.

For the converse, suppose $f \in M_{\omega}^{m}(\alpha, \beta, \lambda, l, b)$. Since

$$|a_k| \leq \frac{b(1-\beta)}{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-\beta)]} \quad k \geq 2$$

and also we may set

$$\mu_k = \frac{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-\beta)]}{b(1-\beta)} |a_k|, \quad k \geq 2$$

and $\mu = 1 - \sum_{k=2}^{\infty} \mu_k$. Then

$$f(z) = (z-\omega) - \sum_{k=2}^{\infty} a_k (z-\omega)^k = (z-\omega) -$$

$$\sum_{k=2}^{\infty} \mu_k \frac{b(1-\beta)}{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-\beta)]} (z-\omega)^k$$

$$= (z-\omega) - \sum_{k=2}^{\infty} \mu_k \big[(z-\omega) - f_k(z)\big] = (z-\omega) - \sum_{k=2}^{\infty} \mu_k (z-\omega) + \sum_{k=2}^{\infty} \mu_k f_k(z)$$

$$= \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$$

Corollary A. The extreme points of $M_\omega^m(\alpha,\beta,\lambda,l,b)$ are the functions $f_1(z)=(z-\omega)$ and

$$(z-\omega)-\frac{b(1-\beta)}{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-\beta)]}(z-\omega)^k \quad k\geq 2$$

Our next results are on radii of close-to-convexity, starlikeness and convexity.

A function $f\in M_\omega^m(\alpha,\beta,\lambda,l,b)$ is said to be ω-close-to-convex of order τ if it satisfies $\mathrm{Re}[f'(z)]>\tau$, $(0\leq\tau<1,\ z\in U,\ \omega$ is an arbitrary fixed point in U).

Also a function $f\in M_\omega^m(\alpha,\beta,\lambda,l,b)$ is said to be ω-convex of order τ if and only if $(z-\omega)f'(z)$ is ω-starlike of order τ, that is if

$$\mathrm{Re}\Big[1+\frac{(z-\omega)f''(z)}{f(z)}\Big]>\tau \quad (0\leq\tau<1,\ z\in U).$$

Theorem 2.6 *Let$f\in M_\omega^m(\alpha,\beta,\lambda,l,b)$. Then f is ω-close-to convex of order τ in $|z-\omega|<R_1$, where*

$$R_1=\inf_{k\geq 2}\left[\frac{(1-\tau)(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}{kb(1-\beta)}\right]^{\frac{1}{k-1}}$$

Proof. It is sufficient to show that $|f'(z)-1|\leq 1-\tau$ for $|z-\omega|<R_1$. We have

$$\big|f'(z)-1\big|=\Big|-\sum_{k=2}^{\infty}ka_k(z-\omega)^{k-1}\Big|\leq\sum_{k=1}^{\infty}ka_k(z-\omega)^{k-1}.$$

Thus $|f'(z)-1|\leq\ 1-\tau$ if

$$\sum_{k=2}^{\infty}\left(\frac{k}{1-\tau}\right)\big|a_k\big|\big|z-\omega\big|^{k-1}\ \leq\ 1 \tag{11}$$

But Theorem 2.1 confirms that

$$\sum_{k=2}^{\infty}\frac{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-\beta)]}{b(1-\beta)}|a_k|\ \leq\ 1. \tag{12}$$

Hence (11) will be true if

$$\frac{k|z-\omega|^{k-1}}{1-\tau}\ \leq\ \frac{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-\beta)]}{b(1-\beta)}.$$

We obtain

$$\big|z-\omega\big|\leq\left[\frac{(1-\tau)(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}{kb(1-\beta)}\right]^{\frac{1}{k-1}}$$

as required.

Theorem 2.7 *Let $f \in M_\omega^m(\alpha,\beta,\lambda,l,\lambda,b)$. Then f is ω-starlike of order τ in $|z-\omega|<R_2$ where*

$$R_2 = \inf_{k\geq 2}\left[\frac{(1-\tau)(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}{b(k-\tau)(1-\beta)}\right]^{\frac{1}{k-1}}$$

Proof. We have to show that $\left|\frac{(z-\omega)f'(z)}{f(z)}-1\right| \leq 1-\tau$ for $|z-\omega|<R_2$.
We have

$$\begin{aligned}\left|\frac{(z-\omega)f'(z)}{f(z)}-1\right| &= \left|\frac{-\sum_{k=2}^{\infty}(k-1)a_k(z-\omega)^{k-1}}{1-\sum_{k=2}^{\infty}a_k(z-\omega)^{k-1}}\right| \\ &\leq \frac{\sum_{k=2}^{\infty}(k-1)|a_k||z-\omega|^{k-1}}{1-\sum_{k=2}^{\infty}|a_k||z-\omega|^{k-1}} \leq 1-\tau \end{aligned} \tag{13}$$

Hence (13) holds true if

$$\sum_{k=2}^{\infty}(k-1)|a_k||z-\omega|^{k-1} \leq (1-\tau)\big[1-\sum_{k=2}^{\infty}|a_k||z-\omega|^{k-1}\big]$$

or equivalently to say

$$\sum_{k=2}^{\infty}\Big(\frac{k-\delta}{1-\delta}\Big)|a_k||z-\omega|^{k-1} \leq 1. \tag{14}$$

Hence, by (12), (14) will be true if

$$\Big(\frac{k-\delta}{1-\delta}\Big)|z-\omega|^{k-1} \leq \frac{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-\beta)]}{b(1-\beta)}$$

or if

$$|z-\omega| \leq \left[\frac{(1-\tau)(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}{b(k-\tau)(1-\beta)}\right]^{\frac{1}{k-1}} \quad k\geq 2$$

which complete the proof.

Theorem 2.8 *Let $f \in M_\omega^m(\alpha,\beta,\lambda,l,b)$. Then f is convex of order τ in $|z-\omega|<R_3$, where*

$$R_3 = \inf_{k\geq 2}\left[\frac{(1-\tau)(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}{b(k-\tau)(1-\beta)}\right]^{\frac{1}{k-1}}.$$

Proof: By using the same method of proof as in Theorem 2.7 we can show that

$$\left|\frac{(z-\omega)f''(z)}{f'(z)}\right| \leq 1-\tau(0\leq\tau<1) for\, |z-\omega| \leq R_3,$$

with the support of Theorem 2.1 Thus we have the required result.

Our next result is on inclusion theorem which involves the modified Hadamard products. For every function of the form

$$f_j(z) = (z-\omega) - \sum_{k=2}^{\infty} |a_{k,j}|\,(z-\omega)^k \quad (j=1,2) \tag{15}$$

in the class $A(\omega)$, we define the ω-modified Hadamard product $(f_1 * f_2)(z)$ of $f_1(z)$ and $f_2(z)$ given by

$$f_1(z) * f_2(z) = (z-\omega) - \sum_{k=2}^{\infty} |a_{k,1}|\,|a_{k,2}|\,(z-\omega)^k$$

With the above statements, we can prove the following:

Theorem 2.9 *Let the functions $f_j(z) \quad (j=1,2)$ given by (15) be in the class $M_\omega^m(\alpha,\beta,\lambda,l,b)$. Then $(f_1 * f_2)(z) \in M_\omega^m(\alpha,\beta,\lambda,l,b)$,*

$$\varphi = 1 - \frac{b(1-\beta)^2(2-\beta)(1+\alpha)}{(r+d)\Psi(2,\lambda,l)(m+1)[2-b\beta+\alpha(2-b)]^2 - b^2(1-\beta)^2}$$

where $\Psi(2,\lambda,l) = \frac{1+\lambda+l}{1+l}$.

Proof: By making use of the technique of Schild and Silverman [12], we need to find the largest φsuch that

$$\sum_{k=0}^{\infty} \frac{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}{b(1-\varphi)}\,|a_{k,1}|\,|a_{k,2}| \leq 1$$

since $f_j(z) \in M_\omega^m(\alpha,\beta,\lambda,l,b)$ (j= 1,2), then we have

$$\sum_{k=0}^{\infty} \frac{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}{b(1-\beta)}\,|a_{k,1}| \leq 1$$

and

$$\sum_{k=0}^{\infty} \frac{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}{b(1-\beta)}\,|a_{k,2}| \leq 1$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{k=0}^{\infty} \frac{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}{b(1-\beta)} \sqrt{|a_{k,1}|\,|a_{k,2}|} \leq 1$$

Thus it is sufficient to show that

$$\sum_{k=0}^{\infty} \frac{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}{b(1-\varphi)} |a_{k,1}|\,|a_{k,2}|$$

$$\leq \sum_{k=0}^{\infty} \frac{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}{b(1-\beta)} \sqrt{|a_{k,1}|\,|a_{k,2}|} \quad (k=2)$$

that is,

$$\sqrt{|a_{k,1}|\,|a_{k,2}|} \leq \frac{(1-\varphi)[k-b\beta+\alpha(k-b)]}{(1-\beta)[k-b\beta+\alpha(k-b)]}$$

note that

$$\sqrt{|a_{k,1}|\,|a_{k,2}|} \leq \frac{b(1-\beta)}{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]}$$

Consequently, we need only to show that

$$\frac{b(1-\beta)}{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]} \leq \frac{(1-\varphi)[k-b\beta+\alpha(k-b)]}{(1-\beta)[k-b\beta+\alpha(k-b)]} \quad (k \geq 2)$$

or, equivalently, that

$$\varphi \leq 1 - \frac{b(1-\beta)^2(k-b)(\alpha+1)}{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]^2 - b^2(1-\beta)^2} \quad (k \geq 2)$$

$$A(k) = 1 - \frac{b(1-\beta)^2(k-b)(\alpha+1)}{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]^2 - b^2(1-\beta)^2} \quad (k \geq 2)$$

is an increasing function $k \quad (k \geq 2)$, letting $k=2$ in last equation, we obtain

$$\varphi \leq A(2) = 1 - \frac{b(1-\beta)^2(k-b)(\alpha+1)}{(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)]^2 - b^2(1-\beta)^2}$$

where $I_1(2,\lambda,l) = \frac{1+\lambda+l}{1+l}$ and this complete the proof.

The next result is on convolution and integral operators. Let $f(z)$ be defined by (2), and suppose that

$$g(z) = (z-\omega) - \sum_{k=2}^{\infty} |c_k| (z-\omega)^k$$

Then, the Hadamard product (or convolution) of $f(z)$ and $g(z)$ defined by

$$f(z) * g(z) = (f*g)(z) = (z-\omega) - \sum_{k=2}^{\infty} |a_k| |c_k| (z-\omega)^k$$

Theorem 2.10. *Let $f(z) \in M_{\omega}^{m}(\alpha, \beta, \lambda, l, b)$, and*

$$g(z) = (z-\omega) - \sum_{k=2}^{\infty} |c_k| (z-\omega)^k \quad (0 \leq |c_k| \leq 1).$$

Then $f*g \in M_{\omega}^{m}(\alpha, \beta, \lambda, l, b)$

Proof: In view of Theorem 2.1, we have

$$\sum_{k=2}^{\infty}(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)] |a_k| |c_k|$$

$$\leq \sum_{k=2}^{\infty}(r+d)^{k-1}\Psi(k,\lambda,l)\Phi(m,k)[k-b\beta+\alpha(k-b)] |a_k| \leq b(1-\beta)$$

Theorem 2.11: *Let $f \in M_{\omega}^{m}(\alpha, \beta, \lambda, l, b)$ and let ρ be real number such that $\rho > -1$, then the function*

$$F^{\omega}(z) = \frac{\rho+1}{(z-\omega)^{\rho}} \int_{\omega}^{z} (t-\omega)^{\rho-1} f(t)dt$$

also belongs to the class $M_{\omega}^{m}(\alpha, \beta, \lambda, l, b)$.

Proof: From the representation of $f(z)$, it follows that

$$F^{\omega}(z) = (z-\omega) - \sum_{k=2}^{\infty} |A_k| (z-\omega)^k$$

where $A_k = \left(\frac{\rho+1}{\rho+k}\right)|a_k|$. Since $\rho > -1$, then$0 \leq A_k \leq |a_k|$. Which in view of Theorem 2.1, $F \in M_{\omega}^{m}(\alpha, \beta, \lambda, l, b)$.

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