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THREE DIMENSIONAL SURFACES FOLIATED BY LORENTZ SPHERES IN E_1^7

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Abstract. In this paper, we study 3-dimensional surfaces in E_1^7 generated by equiform motions of a Lorentz sphere. The properties of these surfaces up to first order are investigated. We show that, as it is shown in E^7, any 3–surface of the studied type in E_1^7 in general is contained in a canal hypersurface, which is gained as envelope of a one-parametric set of 6-dimensional pseudospheres. Finally we give an example.

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1. Introduction

Equiform motions are a general form of Euclidean motions. It is crucial that equiform motions are regular motions. These motions are studied in kinematic and differential geometry in recent years frequently. An equiform transformation in the n-dimensional semi-Euclidean space with the index 1 is an affine transformation whose linear part is composed by an semi orthogonal transformation and a homothetical transformation. Such an equiform transformation maps points $x \in E_1^n$ according to the rule

$$x \to \rho Ax + d, A \in SO_1(n), \rho \in R^+, d \in E_1^n. \tag{1}$$

The number ρ is called the scaling factor. An equiform motion is defined if the parameters of (1), including s, are given as function of a time parameter t. Then a smooth one-parameter equiform motion moves a point x via $x(t) = \rho(t)A(t)x(t) + d(t)$. The kinematic corresponding to this transformation group is called equiform kinematic [1, 6, 9]. The authors give some first order properties of cyclic surfaces generated by equiform motions in five dimensional Euclidean space and semi-Euclidean space [1, 10, 2]. Moreover it is studied 3-dimensional surfaces in E^7 generated by equiform motions of a sphere and prove that 3-dimensional surfaces in E^7, in general, is contained in a canal hypersurfaces [4]. In [6] it is showed that a kinematic three-dimensional surface obtained by the equiform motion of a sphere and with constant scalar curvature K satisfies $|K| < 2$.

In Minkowski space E_1^3 with scalar product $< x, y >= -x_1y_1 + x_2y_2 + x_3y_3$ the pseudosphere or Lorentz sphere and the pseudohyperbolic surface play the same role as spheres in Euclidean space. Lorentz sphere of radius $r > 0$ in E_1^3 is the quadric

$$S_1^2(r) = \{p \in E_1^3 : \langle p, p\rangle = r^2\}.$$

This surface is timelike and is the hyperboloid of one sheet $-x_1^2 + x_2^2 + x_3^2 = r^2$ which is obtained by rotating the hyperbola $-x_1^2 + x_3^2 = r^2$ in the plane $x_2 = 0$ with respect to the x_1 axis. The pseudohyperbolic surface is the quadric

$$H_0^2(r) = \{p \in E_1^3 : \langle p, p\rangle = -r^2\}.$$

This surface is spacelike and is the hyperboloid of two sheet $-x_1^2 + x_2^2 + x_3^2 = -r^2$ which is obtained by rotating the hyperbola $x_1^2 - x_3^2 = r^2$ in the plane $x_2 = 0$ with respect to the x_1 axis [5].

In this paper we consider the equiform motions of a Lorentz sphere k_o in E_1^n. The point paths of the Lorentz sphere generate 3-dimensional surface, containing the positions of the starting Lorentz sphere k_o. We have studied the first order properties of these surfaces for the points of these Lorentz spheres for arbitrary dimensions $n \geq 3$. We restrict our considerations to dimension $n = 7$ because at any moment the infinitesimal transformations of the motion map the points of the Lorentz sphere k_o to the velocity vectors, whose end points will form an affine image of k_o (in general a Lorentz sphere) that span a subspace W of E_1^n with $n \leq 7$. Moreover we show that, as it is shown in E^7, any 3–surface of the studied type in E_1^7 in general is contained in a canal hypersurface, which is gained as envelope of a one-parametric set of 6-dimensional pseudospheres. Finally we give an example.

2.Local study in canonical frames

Consider a unit Lorentz sphere k_o in the space $\pi_o = [x_1x_2x_3]$ centered at the origin represented by $x(\theta, \phi) = (\sinh\theta, \cosh\theta\sin\phi, \cosh\theta\cos\phi, 0, 0, 0, 0)^T$, $\theta \in R$ and $\phi \in [0, 2\pi]$, the general representation of a 3-dimensional surface in E_1^7 is given by

$$X(t, \theta, \phi) = \rho(t)A(t)x(\theta, \phi) + d(t),\ t \in R \tag{2}$$

where $\rho(t)$ is a scaling factor, $A(t) = (a_{ij}(t)) : i, j = 1, 2, \ldots, 7$ is a semi orthogonal matrix and $d(t) = (b_1(t), b_2(t), b_3(t), b_4(t), b_5(t), b_6(t), b_7(t))^T$ is the translational part of the motion. Moreover we assume that the all involved functions are of class C^1. By using Taylor's expansion, up to the first order the representation of motion is given by

$$X(t,\theta,\phi)=[\rho(0)A(0)+(\rho^{'}(0)A(0)+\rho(0)A^{'}(0))t]x(\theta,\phi)+d(0)+d^{'}(0)t,$$

where $(')$ denotes differentiation with respect to time $(t=0)$. We assume the moving frame E_1^7 and fixed frame Σ coinciding at the zero position $(t=0)$, then we have

$$A(0)=I,\quad \rho(0)=1 \ \text{ and } \ d(0)=0.$$

Thus we have

$$X(t,\theta,\phi)=[I_7+(\rho^{'}(0)I_7+A^{'}(0))t]x(\theta,\phi)+d^{'}(0)t$$

where $A^{'}(0)=(\omega_k)$, $k=1,2,\ldots,21$ is a semi skew symmetric matrix. For simplicty we write $\rho^{'}$ and $b_i^{'}$ instead of $\rho^{'}(0)$ and $b_i^{'}(0)$ respectively in the rest of this section and in section 3. In these frames, the representation of the motion up to the first order is given by

$$\begin{aligned}\begin{pmatrix}X_1\\X_2\\X_3\\X_4\\X_5\\X_6\\X_7\end{pmatrix} &= t\begin{pmatrix}b_1^{'}\\b_2^{'}\\b_3^{'}\\b_4^{'}\\b_5^{'}\\b_6^{'}\\b_7^{'}\end{pmatrix}+\sinh\theta\begin{pmatrix}1+t\rho^{'}\\t\omega_1\\t\omega_2\\t\omega_3\\t\omega_4\\t\omega_5\\t\omega_6\end{pmatrix} \\ &\quad +\cosh\theta\sin\phi\begin{pmatrix}t\omega_1\\1+t\rho^{'}\\-t\omega_7\\-t\omega_8\\-t\omega_9\\-t\omega_{10}\\-t\omega_{11}\end{pmatrix}+\cosh\theta\cos\phi\begin{pmatrix}t\omega_2\\t\omega_7\\1+t\rho^{'}\\-t\omega_{12}\\-t\omega_{13}\\-t\omega_{14}\\-t\omega_{15}\end{pmatrix} \\ &= t\overrightarrow{b}+\overrightarrow{a}_0\sinh\theta+\overrightarrow{a}_1\cosh\theta\sin\phi+\overrightarrow{a}_2\cosh\theta\cos\phi\end{aligned}\tag{3}$$

For any fixed t in equation (3), we generally get an elliptical hyperboloid for $\theta\in R$ and $\phi\in[0,2\pi]$ centered at the point $t(b_1^{'},b_2^{'},b_3^{'},b_4^{'},b_5^{'},b_6^{'},b_7^{'})$. The latter elliptical hyperboloid turns to a 2−dimensional Lorentz sphere if $\overrightarrow{a}_0$, $\overrightarrow{a}_1$ and $\overrightarrow{a}_2$ form a orthogonal basis. This gives the conditions

$$\begin{aligned}\omega_2\omega_7+\omega_3\omega_8+\omega_4\omega_9+\omega_5\omega_{10}+\omega_6\omega_{11} &= -\omega_1\omega_7+\omega_3\omega_{12}+\omega_4\omega_{13}+\omega_5\omega_{14}+\omega_6\omega_{15}\\ &= -\omega_1\omega_2+\omega_8\omega_{12}+\omega_9\omega_{13}+\omega_{10}\omega_{14}+\omega_{11}\omega_{15}\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\omega_1^2+\omega_2^2+\omega_3^2+\omega_4^2+\omega_5^2+\omega_6^2 &= \omega_1^2-\omega_7^2-\omega_8^2-\omega_9^2-\omega_{10}^2-\omega_{11}^2\\ &= \omega_2^2-\omega_7^2-\omega_{12}^2-\omega_{13}^2-\omega_{14}^2-\omega_{15}^2\\ &= a,\end{aligned}$$

where $a\in R^+$. Thus we get following equation of the Lorentz sphere

$$\sum_{i=1}^{7}\varepsilon_i(x_i-tb_i^{'})^2=(1+t\rho^{'})^2-at^2.$$

where $\varepsilon_1=-1, \varepsilon_j=1, j=2,3,4,5,6,7$. The orthogonal projection of these elliptical hyperboloid in (3) on the space of the starting Lorentz sphere $\pi_o=[x_1x_2x_3]$ is

$$\begin{aligned}\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix} &= t\begin{pmatrix}b_1^{'}\\b_2^{'}\\b_3^{'}\end{pmatrix}+\sinh\theta\begin{pmatrix}1+t\rho^{'}\\t\omega_1\\t\omega_2\end{pmatrix}\\ &\quad+\cosh\theta\sin\phi\begin{pmatrix}t\omega_1\\1+t\rho^{'}\\-t\omega_7\end{pmatrix}+\cosh\theta\cos\phi\begin{pmatrix}t\omega_2\\t\omega_7\\1+t\rho^{'}\end{pmatrix}.\end{aligned}\tag{4}$$

This equation generalizes in five dimension what happens for $\phi=0$. Namely: if $\phi=0$ the orthogonal projection of the elliptical hyperboloid in (4) on the space $[x_1x_3]$ is

$$\begin{pmatrix}X_1\\X_3\end{pmatrix}=t\begin{pmatrix}b_1^{'}\\b_3^{'}\end{pmatrix}+\sinh\theta\begin{pmatrix}1+t\rho^{'}\\t\omega_2\end{pmatrix}+\cosh\theta\begin{pmatrix}t\omega_2\\1+t\rho^{'}\end{pmatrix}.$$

This gives Lorentzian circles centered at $(tb_1^{'},tb_3^{'})$ and radii by $r=\sqrt{\left|t^2\omega_2^2-(1+t\rho^{'})^2\right|}$. See also [10].

Corollary 1. *The orthogonal projection of the elliptical hyperboloids into the space of the starting Lorentz sphere in general are elliptical hyperboloids for any fixed t, in particular hyperboloids of revolution iff $\omega_2=\omega_7$, centered at $(tb_1^{'},tb_2^{'},tb_3^{'})$.*

The projection of the ruled surface of tangent to k_o into the original space will give a 3-dimensional surface in E_1^3, which is foliated by elliptical hyperboloids. Now from (4) we have

$$X(t,\theta,\phi)=\begin{pmatrix} 1+t\rho' & t\omega_1 & t\omega_2 \\ t\omega_1 & 1+t\rho' & t\omega_7 \\ t\omega_2 & -t\omega_7 & 1+t\rho' \end{pmatrix}\begin{pmatrix} \sinh\theta \\ \cosh\theta\sin\phi \\ \cosh\theta\cos\phi \end{pmatrix}+t\begin{pmatrix} b_1' \\ b_2' \\ b_3' \end{pmatrix}$$

and the first partial derivatives are

$$X_t=\begin{pmatrix} b_1' \\ b_2' \\ b_3' \end{pmatrix}+\begin{pmatrix} \rho' & \omega_1 & \omega_2 \\ \omega_1 & \rho' & \omega_7 \\ \omega_2 & -\omega_7 & \rho' \end{pmatrix}\begin{pmatrix} \sinh\theta \\ \cosh\theta\sin\phi \\ \cosh\theta\cos\phi \end{pmatrix},$$

$$X_\theta=(\cosh\theta,\sinh\theta\sin\phi,\sinh\theta\cos\phi)^T,$$
$$X_\phi=(0,\cosh\theta\cos\phi,-\cosh\theta\sin\phi)^T.$$

Then the linearly dependent points

$$\cosh\theta[\rho'-b_1'\sinh\theta+b_2'\cosh\theta\sin\phi+b_3'\cosh\theta\cos\phi]=0,$$

we get

$$\cosh\theta[\rho'+\langle d',x(\theta,\phi)\rangle]=0.$$

The latter equation characterizes the instantaneous curve of contact.

3.The tangent pseudosphere of 3-dimensional surface in E_1^7

In this section, we will show at any instant t there exist a pseudosphere $K(t)$, which is tangent to a given 3-dimensional surface (2) in all points of the instantaneous position $k(t)$ of the Lorentz sphere k_o. Without loss of generality we investigate the situation at the zero position. Any pseudosphere K_o which is tangent to given 3-dimensional surface (2) along k_o has to contain k_o; hence the center of K_o has coordinates $(0,0,0,m_4,m_5,m_6,m_7)$ with $m_4,m_5,m_6,m_7\in R$. On the other hand since K_o has to be tangent to all velocity vectors of the motion, the center of K_o has to lie in each of the hyperplanes through the points of $k(t)$ orthogonal to these velocity vectors. This gives us the additional condition

$$\begin{aligned} & m_4(b_4'+\omega_3\sinh\theta-\omega_8\cosh\theta\sin\phi-\omega_{12}\cosh\theta\cos\phi) \\ & +m_5(b_5'+\omega_4\sinh\theta-\omega_9\cosh\theta\sin\phi-\omega_{13}\cosh\theta\cos\phi) \\ & +m_6(b_6'+\omega_5\sinh\theta-\omega_{10}\cosh\theta\sin\phi-\omega_{14}\cosh\theta\cos\phi) \\ & +m_7(b_7'+\omega_6\sinh\theta-\omega_{11}\cosh\theta\sin\phi-\omega_{15}\cosh\theta\cos\phi) \\ = & \ \rho'-b_1'\sinh\theta+b_2'\cosh\theta\sin\phi+b_3'\cosh\theta\cos\phi \end{aligned} \tag{5}$$

By comparing the coefficients of $\{1, \sinh\theta, \cosh\theta\sin\phi, \cosh\theta\cos\phi\}$ in (5), we have the system of linear equations

$$BM = H, \tag{6}$$

where

$$B = \begin{pmatrix} b_4^{'} & b_5^{'} & b_6^{'} & b_7^{'} \\ \omega_3 & \omega_4 & \omega_5 & \omega_6 \\ \omega_8 & \omega_9 & \omega_{10} & \omega_{11} \\ \omega_{12} & \omega_{13} & \omega_{14} & \omega_{15} \end{pmatrix}, \ M = \begin{pmatrix} m_4 \\ m_5 \\ m_6 \\ m_7 \end{pmatrix} \text{ and } H = \begin{pmatrix} \rho^{'} \\ -b_1^{'} \\ -b_2^{'} \\ -b_3^{'} \end{pmatrix}.$$

If B is a regular matrix, we get

$$M = B^{-1}H. \tag{7}$$

Therefore, we have the following theorem:

Theorem 1. *In general there is a 6-dimensional pseudosphere with center $(0, 0, 0, m_4, m_5, m_6, m_7)$ which contains the Lorentz sphere k_o and is tangent to all tangent planes $\tau(\theta, \phi)$ of the given 3-dimensional surface (2). This pseudosphere is given by*

$$-x_1^2 + x_2^2 + x_3^2 + (x_4 - m_4)^2 + (x_5 - m_5)^2 + (x_6 - m_6)^2 + (x_7 - m_7)^2 = 1 + \sum_{i=4}^{7} m_i^2,$$

where m_4, m_5, m_6, m_7 given by (7).

Definition 1. *Canal hypersurfaces in E_1^n are envelope hypersurfaces of one-parametric sets of pseudospheres.*

Therefore, we have the following theorem:

Theorem 2. *Any 3-dimensional surface of the studied type in E_1^7 in general is contained in a canal hypersurface, which is gained as envelope of a one-parametric set of 6-dimensional pseudospheres.*

If the system of equations (6) is singular, we have many cases. This situation is the same of the singular cases in [4] and we omit the details.

4. Curve of centers of the pseudospheres

Now, we consider t is varying and in this section, we will determine the centers of the pseudospheres which contain the Lorentz sphere $k(t)$ and are tangent to all tangent planes $\tau(t,\theta,\phi)$ of the 3-dimensional surface (2). If Let $a_i(t), i = 1,\ldots,7$ are the column vectors of the matrix $A(t)$, then (2) can be written as follows

$$X(t,\theta,\phi) = \rho(t)[a_1(t)\sinh\theta + a_2(t)\cosh\theta\sin\phi + a_3(t)\cosh\theta\cos\phi] + d(t) \tag{8}$$

where $d(t)$ is the center of the moving Lorentz sphere and $a_1(t), a_2(t), a_3(t)$ are three orthogonal vectors in the space of the moving Lorentz sphere. The velocity vectors of the points of the Lorentz sphere are given by

$$\begin{aligned} X'(t,\theta,\phi) &= [\rho'(t)a_1(t) + \rho(t)a_1'(t)]\sinh\theta \\ &\quad +[\rho'(t)a_2(t) + \rho(t)a_2'(t)]\cosh\theta\sin\phi \\ &\quad +[\rho'(t)a_3(t) + \rho(t)a_3'(t)]\cosh\theta\cos\phi + d'(t) \end{aligned} \tag{9}$$

where $'$ denotes the derivative with respect to the time t.

The equation of the hyperplanes orthogonal to such a path is

$$Y^T X'(t,\theta,\phi) = X^T(t,\theta,\phi)X'(t,\theta,\phi)$$

where $Y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7)^T$ is the position vector of an arbitrary point Y in the hyperplane. The scalar product in the above equation is Lorentz metric. According to the inner product this equation is

$$Y^T \varepsilon X'(t,\theta,\phi) = X^T(t,\theta,\phi)\varepsilon X'(t,\theta,\phi) \tag{10}$$

where $\varepsilon = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ is the sign matrix.

Subsitution of (8) and (9) into (10), we have

$$\begin{aligned} & Y^T\varepsilon[\rho'(t)a_1(t) + \rho(t)a_1'(t)]\sinh\theta + Y^T\varepsilon[\rho'(t)a_2(t) + \rho(t)a_2'(t)]\cosh\theta\sin\phi \\ & +Y^T\varepsilon[\rho'(t)a_3(t) + \rho(t)a_3'(t)]\cosh\theta\cos\phi + Y^T\varepsilon d'(t) \\ = & \ \big(\rho(t)a_1^T(t)\sinh\theta + \rho(t)a_2^T(t)\cosh\theta\sin\phi + \rho(t)a_3^T(t)\cosh\theta\cos\phi + d(t)\big)\,\varepsilon \\ & ([\rho'(t)a_1(t) + \rho(t)a_1'(t)]\sinh\theta + [\rho'(t)a_2(t) + \rho(t)a_2'(t)]\cosh\theta\sin\phi \\ & +[\rho'(t)a_3(t) + \rho(t)a_3'(t)]\cosh\theta\cos\phi + d'(t)). \end{aligned} \tag{11}$$

Since $A^T\varepsilon A=\varepsilon$ and $A^T\varepsilon A'$ is a skew symmetric matrix, let $e_k(t)=a_k^T(t)\varepsilon d'(t)$, $h_k(t)=d^T(t)\varepsilon a_k'(t)$ and $l_k(t)=d^T(t)\varepsilon a_k(t)$, $k=1,2,3$. Then by comparing the coefficients of $\{1,\sinh\theta,\cosh\theta\sin\phi,\cosh\theta\cos\phi\}$ in (11), we obtain

$$\sum_{i=1}^{7}\varepsilon_i y_i b_i'(t)=\sum_{i=1}^{7}\varepsilon_i b_i(t)b_i'(t)+\rho(t)\rho'(t)$$

$$\rho'(t)\sum_{i=1}^{7}\varepsilon_i y_i a_{i1}(t)+\rho(t)\sum_{i=1}^{7}\varepsilon_i y_i a_{i1}'(t)=\rho(t)(e_1(t)+h_1(t))+\rho'(t)l_1(t) \tag{12}$$

$$\rho'(t)\sum_{i=1}^{7}\varepsilon_i y_i a_{i2}(t)+\rho(t)\sum_{i=1}^{7}\varepsilon_i y_i a_{i2}'(t)=\rho(t)(e_2(t)+h_2(t))+\rho'(t)l_2(t)$$

$$\rho'(t)\sum_{i=1}^{7}\varepsilon_i y_i a_{i3}(t)+\rho(t)\sum_{i=1}^{7}\varepsilon_i y_i a_{i3}'(t)=\rho(t)(e_3(t)+h_3(t))+\rho'(t)l_3(t)$$

where $\varepsilon_1=-1,\varepsilon_j=1,\ j=2,3,4,5,6,7.$

We know from the inital position, that the hyperplanes of the 3-dimensional surfaces contain a point $m(t)$ for any t and $\forall\theta,\phi$ such that $m(t)=(0,0,0,m_4(t),m_5(t),m_6(t),m_7(t))$ is the center of this pseudosphere, then from (12), one can find

$$FM=Q, \tag{13}$$

where

$$F=\begin{pmatrix} b_4'(t) & b_5'(t) & b_6'(t) & b_7'(t)\\ T_{41}(t) & T_{51}(t) & T_{61}(t) & T_{71}(t)\\ T_{42}(t) & T_{52}(t) & T_{62}(t) & T_{72}(t)\\ T_{43}(t) & T_{53}(t) & T_{63}(t) & T_{73}(t)\end{pmatrix},\qquad M=\begin{pmatrix} m_4(t)\\ m_5(t)\\ m_6(t)\\ m_7(t)\end{pmatrix}$$

and

$$Q=\begin{pmatrix} \sum\limits_{i=1}^{7}\varepsilon_i b_i(t)b_i'(t)+\rho(t)\rho'(t)\\ \rho(t)(e_1(t)+h_1(t))+\rho'(t)l_1(t)\\ \rho(t)(e_2(t)+h_2(t))+\rho'(t)l_2(t)\\ \rho(t)(e_3(t)+h_3(t))+\rho'(t)l_3(t)\end{pmatrix}$$

where $T_{kr}(t)=\rho'(t)a_{kr}(t)+\rho(t)a_{kr}'(t)$, $k=4,5,6,7,\ r=1,2,3.$

If F is a regular matrix, we obtain

$$M=F^{-1}Q \tag{14}$$

Therefore, the coordinates of centers of the Lorentz spheres in the fixed frame at any instant t are given by

$$\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \end{pmatrix} = \rho(t)A(t)\begin{pmatrix} 0 \\ 0 \\ 0 \\ m_4(t) \\ m_5(t) \\ m_6(t) \\ m_7(t) \end{pmatrix} + d(t) \tag{15}$$

Theorem 3. *At any instant t, there is a pseudospheres $K(t)$ with centers given by $(0,0,0,m_4(t),m_5(t),m_6(t),m_7(t))$ which contains the Lorentz sphere $k(t)$, which is tangent to all tangent planes $\tau(t,\theta,\phi)$ of the given 3-dimensional surface (2). The curve of the centers of these pseudospheres in the moving frame is given by $m(t) = (0,0,0,m_4(t),m_5(t),m_6(t),m_7(t))$, where $m_4(t),m_5(t),m_6(t),m_7(t)$ are given by the equation (14) and in the fixed frame its given by (15).*

Example 1. We consider a 3-dimensional surfaces generated by the motion given by

$$A(t) = \begin{pmatrix} \cosh\lambda t & 0 & 0 & 0 & 0 & \sin t\sinh\lambda t & -\cos t\sinh\lambda t \\ 0 & \cos\lambda t & 0 & 0 & \sin\lambda t & 0 & 0 \\ 0 & 0 & \cos\lambda t & \sin\lambda t & 0 & 0 & 0 \\ 0 & 0 & -\sin\lambda t & \cos\lambda t & 0 & 0 & 0 \\ 0 & -\sin\lambda t & 0 & 0 & \cos\lambda t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos t & \sin t \\ -\sinh\lambda t & 0 & 0 & 0 & 0 & -\sin t\cosh\lambda t & \cos t\cosh\lambda t \end{pmatrix} \tag{16}$$

such that $\lambda \in R - \{0\}$. We assume $\rho(t) = e^{qt}$ and $d(t) = (0,0,0,0,0,\nu t,0)^T$, where $q \neq 0$ and $\nu \neq 0$. We compute by differentianting $A(t)$ and put $t = 0$, one can find

$$\begin{aligned} \omega_6 &= -\lambda, \omega_9 = \omega_{12} = \lambda,\ \omega_{21} = 1 \text{ and} \\ \omega_k &= 0,\ k = 1,2,3,4,5,7,8,9,10,11,13,14,15,16,17,18,19,20. \end{aligned}$$

Substutiting into (7), we have

$$m_4 = 0, m_5 = 0,\ m_6 = \frac{q}{\nu},\ m_7 = 0.$$

Then, the pseudosphere which contains a Lorentz sphere k_0 and is tangent to all tangent planes of the corresponding 3-dimensional surface is given by

$$-x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + (x_6 - \frac{q}{\nu})^2 + x_7^2 = 1 + \frac{q^2}{\nu^2}.$$

After differentiation of (16), and substitution into (14), we get

$$m_4(t) = 0, m_5(t) = 0,\ m_6(t) = \frac{\nu^2 t + q e^{2qt}}{\nu},\ m_7(t) = 0.$$

Therefore, the parametric representation of the curve of centers of the pseudospheres in the moving frame is given by

$$m(t) = (0, 0, 0, 0, 0, \frac{\nu^2 t + q e^{2qt}}{\nu}, 0).$$

From (15) and (16) one can see that the parametrization of the curve center in the fixed frame is

$$M(t) = e^{qt} \begin{pmatrix} m_6(t) \sin t \sinh \lambda t \\ 0 \\ 0 \\ 0 \\ 0 \\ m_6(t) \cos t + \nu t \\ -m_6(t) \sin t \cosh \lambda t \end{pmatrix}^T,\ m_6(t) = \frac{\nu^2 t + q e^{2qt}}{\nu}.$$

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