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CONSTRUCTION OF FOCAL CURVES OF TIMELIKE BIHARMONIC GENERAL HELICES IN THE LORENTZIAN E(1,1)

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Abstract. In this paper, we study focal curves of timelike biharmonic general helices in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Finally, we find out their explicit parametric equations.

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1. Introduction

Let $\mathbb{E}(1,1)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Topologically, $\mathbb{E}(1,1)$ is diffeomorphic to $\mathbb{R}^3$ under the map

$$\mathbb{E}(1,1) \longrightarrow \mathbb{R}^3 : \begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow (x,y,z),$$

It's Lie algebra has a basis consisting of

$$\mathbf{e}_1 = \frac{\partial}{\partial x},\ \mathbf{e}_2 = \cosh x\frac{\partial}{\partial y} + \sinh x\frac{\partial}{\partial z},\ \mathbf{e}_3 = \sinh x\frac{\partial}{\partial y} + \cosh x\frac{\partial}{\partial z},$$

for which

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3,\ \ [\mathbf{e}_2, \mathbf{e}_3] = 0,\ \ [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_2.$$

Put

$$x^1 = x,\ x^2 = \frac{1}{2}(y+z),\ x^3 = \frac{1}{2}(y-z).$$

Then, we get

$$\mathbf{e}_1=\frac{\partial}{\partial x^1},\ \mathbf{e}_2=\frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2}+e^{-x^1}\frac{\partial}{\partial x^3}\right),\ \mathbf{e}_3=\frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2}-e^{-x^1}\frac{\partial}{\partial x^3}\right). \tag{1}$$

The bracket relations are

$$[\mathbf{e}_1,\mathbf{e}_2]=\mathbf{e}_3,\ \ [\mathbf{e}_2,\mathbf{e}_3]=0,\ \ [\mathbf{e}_1,\mathbf{e}_3]=\mathbf{e}_2.$$

We consider left-invariant Lorentzian metrics which has a pseudo-orthonormal basis $\{\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3\}$. We consider left-invariant Lorentzian metric [9], given by

$$g=-\left(dx^1\right)^2+\left(e^{-x^1}dx^2+e^{x^1}dx^3\right)^2+\left(e^{-x^1}dx^2-e^{x^1}dx^3\right)^2,$$

where

$$g\left(\mathbf{e}_1,\mathbf{e}_1\right)=-1,\ g\left(\mathbf{e}_2,\mathbf{e}_2\right)=g\left(\mathbf{e}_3,\mathbf{e}_3\right)=1.$$

Let coframe of our frame be defined by

$$\theta^1=dx^1,\ \theta^2=e^{-x^1}dx^2+e^{x^1}dx^3,\ \theta^3=e^{-x^1}dx^2-e^{x^1}dx^3.$$

2.Timelike Biharmonic General Helices in the Lorentzian Group of Rigid Motions $\mathbb{E}(1,1)$

Let $\gamma:I\longrightarrow\mathbb{E}(1,1)$ be a non geodesic timelike curve in the group of rigid motions $\mathbb{E}(1,1)$ parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the group of rigid motions $\mathbb{E}(1,1)$ along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ) and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T}&=\kappa\mathbf{N},\\ \nabla_{\mathbf{T}}\mathbf{N}&=\kappa\mathbf{T}+\tau\mathbf{B},\\ \nabla_{\mathbf{T}}\mathbf{B}&=-\tau\mathbf{N},\end{aligned} \tag{2}$$

where κ is the curvature of γ, τ is its torsion and

$$\begin{aligned}g\left(\mathbf{T},\mathbf{T}\right)&=-1,\ g\left(\mathbf{N},\mathbf{N}\right)=1,\ g\left(\mathbf{B},\mathbf{B}\right)=1,\\ g\left(\mathbf{T},\mathbf{N}\right)&=g\left(\mathbf{T},\mathbf{B}\right)=g\left(\mathbf{N},\mathbf{B}\right)=0.\end{aligned} \tag{3}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{4}$$

Theorem 2.1. ([8]) *$\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a non geodesic timelike biharmonic curve in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$ if and only if*

$$\begin{aligned} \kappa &= \text{constant} \neq 0, \\ \kappa^2 - \tau^2 &= 1 + 2B_1^2, \\ \tau' &= -2N_1B_1. \end{aligned} \tag{5}$$

Theorem 2.2. ([8]) *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a non geodesic timelike biharmonic general helix in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Then, the parametric equations of γ are*

$$\begin{aligned} x^1(s) &= \cosh\eth\kappa s + \wp_3, \\ x^2(s) &= \frac{\sinh\eth e^{\cosh\eth\kappa s+\wp_3}}{2\left(\wp_1^2+\cosh^2\eth\right)}\{(\cosh\eth-\wp_1)\cos(\wp_1\kappa s+\wp_2) \\ &\quad + (\cosh\eth+\wp_1)\sin(\wp_1\kappa s+\wp_2)\} + \wp_4, \\ x^3(s) &= \frac{\sinh\eth e^{-\cosh\eth\kappa s-\wp_3}}{2\left(\wp_1^2+\sinh^2\eth\right)}\{-(\cosh\eth-\wp_1)\cos(\wp_1\kappa s+\wp_2) \\ &\quad + (\cosh\eth+\wp_1)\sin(\wp_1\kappa s+\wp_2)\} + \wp_5, \end{aligned} \tag{6}$$

where $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4$, $\wp_5$ are constants of integration.

3. Focal Curves of Timelike Biharmonic General Helices in the Lorentzian $\mathbb{E}(1,1)$

Denoting the focal curve by $\mathfrak{focal}_\gamma$, we can write

$$\mathfrak{focal}_\gamma(s) = (\gamma + c_1\mathbf{N} + c_2\mathbf{B})(s), \tag{7}$$

where the coefficients $\mathfrak{f}_1$, $\mathfrak{f}_2$ are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively. Further, the focal curvatures $\mathfrak{f}_1$, $\mathfrak{f}_2$ are defined by

$$\mathfrak{f}_1 = -\frac{1}{\kappa},\ \mathfrak{f}_2 = \frac{\mathfrak{f}_1'}{\tau},\ \kappa \neq 0,\ \tau \neq 0. \tag{8}$$

Lemma 3.1. *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a unit speed timelike biharmonic general helix and $\mathfrak{focal}_\gamma$ its focal curve on $\mathbb{E}(1,1)$. Then,*

$$\mathfrak{f}_1 = -\frac{1}{\kappa} = \text{constant and } \mathfrak{f}_2 = 0. \tag{3.3}$$

Proof. Using (2.3) and (3.2), we get (3.3). Thus, the proof is completed.

Lemma 3.2. *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a unit speed timelike biharmonic curve according to flat metric and C_γ its focal curve on $\mathbb{E}(1,1)$. Then,*

$$\mathfrak{focal}_\gamma(s) = (\gamma + \mathfrak{f}_1\mathbf{N})(s). \tag{3.4}$$

Theorem 3.3. *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a unit speed timelike biharmonic general helix and $\mathfrak{focal}_\gamma$ its focal curve on $\mathbb{E}(1,1)$. Then, the equation of $\mathfrak{focal}_\gamma$ is*

$$\begin{aligned}
\mathfrak{focal}_\gamma(s) \quad = \quad & (\cosh\partial\kappa s + \wp_3 - \frac{2\mathfrak{f}_1}{\kappa}\left(\sinh^2\partial\cos(\wp_1\kappa s+\wp_2)\sin(\wp_1\kappa s+\wp_2)\right))\mathbf{e}_1 \\
& +[\frac{\sinh\partial}{2\left(\wp_1^2+\cosh^2\partial\right)}\{(\cosh\partial-\wp_1)\cos(\wp_1\kappa s+\wp_2) \\
& +(\cosh\partial+\wp_1)\sin(\wp_1\kappa s+\wp_2)\}+\wp_4 e^{-\cosh\partial\kappa s-\wp_3} \\
& +\frac{\sinh\partial}{2\left(\wp_1^2+\cosh^2\partial\right)}\{-(\cosh\partial-\wp_1)\cos(\wp_1\kappa s+\wp_2) \\
& +(\cosh\partial+\wp_1)\sin(\wp_1\kappa s+\wp_2)\}+\wp_5 e^{\cosh\partial\kappa s+\wp_3} \\
& -\frac{\mathfrak{f}_1}{\kappa}\sinh\partial\sin(\wp_1\kappa s+\wp_2)\left(\frac{1}{\wp_1\kappa}+\cosh\partial\right)]\mathbf{e}_2 \\
& +[\frac{\sinh\partial}{2\left(\wp_1^2+\cosh^2\partial\right)}\{(\cosh\partial-\wp_1)\cos(\wp_1\kappa s+\wp_2) \\
& +(\cosh\partial+\wp_1)\sin(\wp_1\kappa s+\wp_2)\}+\wp_4 e^{-\cosh\partial\kappa s-\wp_3} \\
& -\frac{\sinh\partial}{2\left(\wp_1^2+\cosh^2\partial\right)}\{-(\cosh\partial-\wp_1)\cos(\wp_1\kappa s+\wp_2) \\
& +(\cosh\partial+\wp_1)\sin(\wp_1\kappa s+\wp_2)\}-\wp_5 e^{\cosh\partial\kappa s+\wp_3} \\
& +\frac{\mathfrak{f}_1}{\kappa}\sinh\partial\cos(\wp_1\kappa s+\wp_2)\left(\frac{1}{\wp_1\kappa}-\cosh\partial\right)]\mathbf{e}_3,
\end{aligned} \tag{9}$$

where $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4$, $\wp_5$ *are constants of integration.*

Proof. First equation of (2.3) and basis, we have

$$\nabla_{\mathbf{T}}\mathbf{T} = \left(T_1' - 2T_2T_3\right)\mathbf{e}_1 + \left(T_2' - T_1T_3\right)\mathbf{e}_2 + \left(T_3' - T_1T_2\right)\mathbf{e}_3. \tag{10}$$

Hence, we express

$$\begin{aligned} \mathbf{N} \;\; = \;\; & -\frac{2}{\kappa}\left(\sinh^2\eth\cos\left(\wp_1\kappa s+\wp_2\right)\sin\left(\wp_1\kappa s+\wp_2\right)\right)\mathbf{e}_1 \\ & -\frac{1}{\kappa}\sinh\eth\sin\left(\wp_1\kappa s+\wp_2\right)\left(\frac{1}{\wp_1\kappa}+\cosh\eth\right)\mathbf{e}_2 \\ & +\frac{1}{\kappa}\sinh\eth\cos\left(\wp_1\kappa s+\wp_2\right)\left(\frac{1}{\wp_1\kappa}-\cosh\eth\right)\mathbf{e}_3. \end{aligned} \tag{11}$$

From (3.4) and (3.7), by direct calculation we have (3.5), which proves the theorem.

Using Theorem 3.3, we can give parametric equations of this curve.

Theorem 3.4. *Let* $\gamma : I \longrightarrow \mathbb{E}(1,1)$ *be a unit speed timelike biharmonic general helix and* $\mathfrak{focal}_\gamma$ *its focal curve on* $\mathbb{E}(1,1)$*. Then, the parametric equation of* $\mathfrak{focal}_\gamma$ *are*

$$x^1_{\mathfrak{focal}_\gamma}(s) = \cosh\eth\kappa s + \wp_3 - \frac{2\mathfrak{f}_1}{\kappa}\left(\sinh^2\eth\cos\left(\wp_1\kappa s+\wp_2\right)\sin\left(\wp_1\kappa s+\wp_2\right)\right),$$

$$
\begin{aligned}
x^2_{\mathfrak{focal}_\gamma}(s) \;=\; & \frac{1}{2}\exp(\cosh\eth\kappa s+\wp_3-\frac{2\lambda}{\kappa}\left(\sinh^2\eth\cos\left(\wp_1\kappa s+\wp_2\right)\sin\left(\wp_1\kappa s+\wp_2\right)\right)) \\
& [[\frac{\sinh\eth}{2\left(\wp_1^2+\cosh^2\eth\right)}\{(\cosh\eth-\wp_1)\cos\left(\wp_1\kappa s+\wp_2\right) \\
& +(\cosh\eth+\wp_1)\sin\left(\wp_1\kappa s+\wp_2\right)\}+\wp_4 e^{-\cosh\eth\kappa s-\wp_3} \\
& +\frac{\sinh\eth}{2\left(\wp_1^2+\cosh^2\eth\right)}\{-(\cosh\eth-\wp_1)\cos\left(\wp_1\kappa s+\wp_2\right) \\
& +(\cosh\eth+\wp_1)\sin\left(\wp_1\kappa s+\wp_2\right)\}+\wp_5 e^{\cosh\eth\kappa s+\wp_3} \\
& -\frac{\mathfrak{f}_1}{\kappa}\sinh\eth\sin\left(\wp_1\kappa s+\wp_2\right)\left(\frac{1}{\wp_1\kappa}+\cosh\eth\right)] \\
& +[\frac{\sinh\eth}{2\left(\wp_1^2+\cosh^2\eth\right)}\{(\cosh\eth-\wp_1)\cos\left(\wp_1\kappa s+\wp_2\right) \\
& +(\cosh\eth+\wp_1)\sin\left(\wp_1\kappa s+\wp_2\right)\}+\wp_4 e^{-\cosh\eth\kappa s-\wp_3} \\
& -\frac{\sinh\eth}{2\left(\wp_1^2+\cosh^2\eth\right)}\{-(\cosh\eth-\wp_1)\cos\left(\wp_1\kappa s+\wp_2\right) \\
& +(\cosh\eth+\wp_1)\sin\left(\wp_1\kappa s+\wp_2\right)\}-\wp_5 e^{\cosh\eth\kappa s+\wp_3} \\
& +\frac{\mathfrak{f}_1}{\kappa}\sinh\eth\cos\left(\wp_1\kappa s+\wp_2\right)\left(\frac{1}{\wp_1\kappa}-\cosh\eth\right)]],
\end{aligned}
$$

$$
\begin{aligned}
x^3_{\mathfrak{focal}_\gamma}(s) &= \frac{1}{2}\exp(-\cosh\partial\kappa s-\wp_3+\frac{2\mathfrak{f}_1}{\kappa}\left(\sinh^2\partial\cos\left(\wp_1\kappa s+\wp_2\right)\sin\left(\wp_1\kappa s+\wp_2\right)\right)) \\
&[[\frac{\sinh\partial}{2\left(\wp_1^2+\cosh^2\partial\right)}\{(\cosh\partial-\wp_1)\cos\left(\wp_1\kappa s+\wp_2\right) \\
&+(\cosh\partial+\wp_1)\sin\left(\wp_1\kappa s+\wp_2\right)\}+\wp_4 e^{-\cosh\partial\kappa s-\wp_3} \\
&+\frac{\sinh\partial}{2\left(\wp_1^2+\cosh^2\partial\right)}\{-(\cosh\partial-\wp_1)\cos\left(\wp_1\kappa s+\wp_2\right) \\
&+(\cosh\partial+\wp_1)\sin\left(\wp_1\kappa s+\wp_2\right)\}+\wp_5 e^{\cosh\partial\kappa s+\wp_3} \\
&-\frac{\mathfrak{f}_1}{\kappa}\sinh\partial\sin\left(\wp_1\kappa s+\wp_2\right)\left(\frac{1}{\wp_1\kappa}+\cosh\partial\right)]\mathbf{e}_2 \\
&-[\frac{\sinh\partial}{2\left(\wp_1^2+\cosh^2\partial\right)}\{(\cosh\partial-\wp_1)\cos\left(\wp_1\kappa s+\wp_2\right) \\
&+(\cosh\partial+\wp_1)\sin\left(\wp_1\kappa s+\wp_2\right)\}+\wp_4 e^{-\cosh\partial\kappa s-\wp_3} \\
&-\frac{\sinh\partial}{2\left(\wp_1^2+\cosh^2\partial\right)}\{-(\cosh\partial-\wp_1)\cos\left(\wp_1\kappa s+\wp_2\right) \\
&+(\cosh\partial+\wp_1)\sin\left(\wp_1\kappa s+\wp_2\right)\}-\wp_5 e^{\cosh\partial\kappa s+\wp_3} \\
&+\frac{\mathfrak{f}_1}{\kappa}\sinh\partial\cos\left(\wp_1\kappa s+\wp_2\right)\left(\frac{1}{\wp_1\kappa}-\cosh\partial\right)]],
\end{aligned}
$$

where $\wp_1,\ \wp_2,\ \wp_3,\ \wp_4,\ \wp_5$ *are constants of integration.*

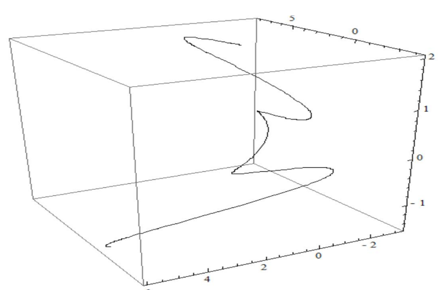

Fig. 1. Biharmonic timelike helix.

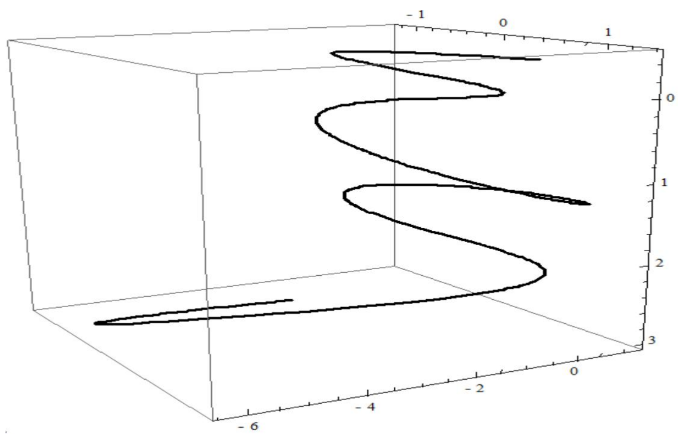

Fig.2. Focal curve of Biharmonic timelike helix.

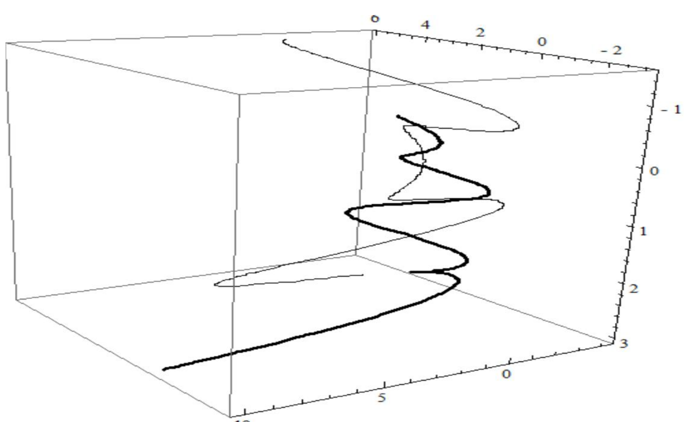

Fig.3. Focal curve and Biharmonic timelike helix.

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