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MODIFICATION OF TRUNCATED EXPANSION METHOD TO SOME COMPLEX NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. Modification of truncated expansion method is used to construct exact solutions of Eckhaus equation, nonlinear Schrödinger equation and Higgs field equation. Modification of truncated expansion method is a powerful solution method for obtaining exact solutions of nonlinear evolution equations.

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1. Introduction

During the past two decades, much effort has been spent on searching for exact solutions of nonlinear equations due to their importance in understanding the nonlinear phenomena. In order to achieve this goal, various direct methods have been proposed, such as tanh method [1,2], multiple exp-function method [3], Backlund transformation method [4], Hirotas direct method [5,6], transformed rational function method [7], first integral method [8-12], simplest equation method [13], an automated tanh-function method [14], Modification of truncated expansion method [15].

In [16], Ma and Chen is used Direct search method to obtain exact solutions of the nonlinear Schrödinger equation in the form

$$i\Psi_t + \Psi_{xx} + \mu|\Psi|^2\Psi = 0, \tag{1}$$

where μ is a real parameter and Ψ is a complex-valued function of the spatial coordinate x and time t.

The Eckhaus equation is in the following form:

$$i\Psi_t + \Psi_{xx} + 2(|\Psi|^2)_x\Psi + |\Psi|^4\Psi = 0, \tag{2}$$

where $\Psi = \Psi(x,t)$ is a complex-valued function of two real variables x, t. This equation is of nonlinear Schrödinger type. The Eckhaus equation was found in

[17] as an asymptotic multiscale reduction of certain classes of nonlinear partial differential equations. In [18], many of the properties of the Eckhaus equation were investigated. In [19], the Eckaus equation was (exactly) linearized by a change (dependent) of variable.
The Higgs field equation [20]

$$u_{tt} - u_{xx} - \alpha u + \beta |u|^2 u - 2uv = 0, \quad (3)$$

$$v_{tt} + v_{xx} - \beta(|u|^2)_{xx} = 0,$$

describes a system of conserved scalar nucleons interacting with a neutral scalar meson. Here, real constant v represents a complex scalar nucleon field and u a real scalar meson field. Eq. (3) is the coupled nonlinear Klein-Gordon equation for $\alpha < 0,\ \beta < 0$ and the coupled Higgs field equation for $\alpha > 0,\ \beta > 0$. The existence of N-soliton solutions for Eq. (3) has been shown by Hirota's bilinear method [21].
The aim of this paper is to find exact solutions of the Eckhaus equation, nonlinear Schrödinger equation and Higgs field equation by using modification of truncated expansion method [15].

2. Modification of truncated expansion method

Let us present the modification of the truncated expansion method [15]. We consider a general nonlinear partial differential equation (PDE) in the form

$$F(u, u_x, u_t, u_{xx}, u_{xt}, ...) = 0. \quad (4)$$

Using traveling wave $u(x,t) = y(z),\ \ z = x - ct$ carries (4) into the following ordinary differential equation (ODE):

$$G(y, y_z, y_{zz}, ...) = 0. \quad (5)$$

The modification of the truncated expansion method contains the following steps [15].
Step 1. Determination of the dominant term with highest order of singularity. To find dominant terms we substitute

$$y = z^{-p}, \quad (6)$$

into all terms of Eq. (5). Then we should compare degrees of all terms of Eq. (5) and choose two or more with the lowest degree. The maximum value of p is the pole of Eq. (5) and we denote it as N. It should be noted that method can be applied when N is integer. If the value N is noninteger one can transform the equation studied.

Step 2. We look for exact solution of Eq. (5) in the form

$$y = \sum_{i=0}^{N} a_i Q^i(z), \tag{7}$$

where $Q(z)$ is the following function

$$Q(z) = \frac{1}{1+e^z}. \tag{8}$$

Step 3. We can calculate necessary number of derivatives of function y. It is easy to do using Maple or Mathematica package. Using case $N = 2$ we have some derivatives of function $y(z)$ in the form

$$\begin{aligned} y &= a_0 + a_1 Q + a_2 Q^2, \\ y_z &= -a_1 Q + (a_1 - 2a_2)Q^2 + 2a_2 Q^3, \\ y_{zz} &= a_1 Q + (4a_2 - 3a_1)Q^2 + (2a_1 - 10a_2)Q^3 + 6a_2 Q^4. \end{aligned} \tag{9}$$

Step 4. We substitute expressions (7)-(9) to Eq. (5). Then we collect all terms with the same powers of function $Q(z)$ and equate expressions to zero. As a result we obtain algebraic system of equations. Solving this system we get the values of unknown parameters.

3. Nonlinear Schrödinger equation

Let us consider the Eq. (1). Substituting $\Psi(x,t) = e^{i(\alpha x + \beta t)} y(z), \quad z = k(x - 2\alpha t)$ into Eq. (1), we obtain ordinary differential equation:

$$-(\beta + \alpha^2)y + k^2 y_{zz} + \mu y^3 = 0. \tag{10}$$

The pole order of Eq. (10) is $N = 1$. So we look for solution of Eq. (10) in the following form

$$y = a_0 + a_1 Q. \tag{11}$$

Substituting (11) into Eq. (10), we obtain the system of algebraic equations in the form

$$\begin{aligned} 2k^2 a_1 + \mu a_1^3 &= 0, \\ -3k^2 a_1 + 3\mu a_0 a_1^2 &= 0, \\ k^2 a_1 - (\beta + \alpha^2)a_1 + 3\mu a_0^2 a_1 &= 0, \end{aligned} \tag{12}$$

$$-(\beta+\alpha^2)a_0+\mu a_0^3=0.$$

Solving these algebraic equations by either Maple or Mathematica, we get

$$a_0=\pm\frac{k\sqrt{-2\mu}}{2\mu},\quad a_1=\pm\frac{2k}{\sqrt{-2\mu}},\quad \beta=-\frac{k^2}{2}-\alpha^2, \tag{13}$$

where α, k are arbitrary constants.
Using the conditions (13) in (11), we obtain

$$y(z)=\pm\frac{k}{2}\sqrt{-\frac{2}{\mu}}(1-2Q(z)). \tag{14}$$

Combining (14) with (8), we obtain the exact solution to Eq. (10) and the exact solution to nonlinear Schrödinger equation can be written as

$$\Psi(x,t)=\pm\frac{k}{2}\sqrt{-\frac{2}{\mu}}e^{i(\alpha x-(\frac{k^2}{2}+\alpha^2)t)}\tanh(\frac{k}{2}(x-2\alpha t)). \tag{15}$$

4. Eckaus equation

In this section we study the Eckaus equation [18]. We may choose the following traveling wave transformation:

$$\Psi(x,t)=e^{i(\alpha x+\beta t)}u(z), \tag{16}$$

where $z=k(x-2\alpha t)$, k, α and β are constants to be determined later. Eq. (2) becomes

$$k^2u_{zz}-(\beta+\alpha^2)u+4ku_zu^2+u^5=0. \tag{17}$$

The pole order of Eq. (17) is $N=\frac{1}{2}$. To obtain an analytic solution, N should be an integer. This requires the use of the transformation

$$u=v^{\frac{1}{2}} \tag{18}$$

that transforms (17) to

$$2k^2vv_{zz}+8kv^2v_z-k^2(v_z)^2-4(\beta+\alpha^2)v^2+4v^4=0. \tag{19}$$

The pole order of Eq. (19) is $N=1$. So we look for solution of Eq. (19) in the following form

$$v=a_0+a_1Q. \tag{20}$$

We substitute Eq. (20) into Eq. (19) and collect all terms with the same power in Q^i $(i=0,1,2,...)$. Equating each coefficient of the polynomial to zero yields a set of simultaneous algebraic equations omitted here for the sake of brevity. Solving these algebraic equations by either Maple or Mathematica, we obtian

Case A.

$$a_0 = 0, \quad a_1 = -\frac{k}{2}, \quad \beta = \frac{k^2}{4} - \alpha^2, \tag{21}$$

where k, α are arbitrary constants.
In this case the exact solution takes the following form:

$$\Psi(x,t) = i\sqrt{\frac{k}{2}} e^{i(\alpha x + (\frac{k^2-4\alpha^2}{4})t)} (\frac{1}{1+e^{k(x-2\alpha t)}})^{\frac{1}{2}}. \tag{22}$$

Case B.

$$a_0 = \frac{k}{2}, \quad a_1 = -\frac{k}{2}, \quad \beta = \frac{k^2}{4} - \alpha^2, \tag{23}$$

where k, α are arbitrary constants.
In this case the exact solution takes the following form:

$$\Psi(x,t) = \sqrt{\frac{k}{2}} e^{i(\alpha x + (\frac{k^2-4\alpha^2}{4})t)} (\frac{1}{2} - \frac{1}{2}\tanh(\frac{k}{2}(x-2\alpha t)))^{\frac{1}{2}}. \tag{24}$$

5. Coupled Higgs field equation

To find exact solutions of coupled Higgs field equation (3), first we make the transformation

$$u(x,t) = e^{i\theta} y(z), \quad v(x,t) = h(z), \tag{25}$$

where $\theta = kx + \omega t,\ z = x + ct$, we have a relation $k = \omega c$ and reduce system (3) to the following system of ordinary differential equations

$$(\omega^2(c^2-1) - \alpha)y + (c^2-1)y_{zz} + \beta y^3 - 2yh = 0, \tag{26}$$

$$(c^2+1)h_{zz} - \beta(y^2)_{zz} = 0.$$

$y(z) = z^{-p_1}$ and $h(z) = z^{-p_2}$ into all terms of Eq. (26). Then we should compare degrees of all terms of Eq. (26) and choose two or more with the lowest degree we have

$$y(z) = a_0 + a_1 Q(z), \quad h(z) = b_0 + b_1 Q(z) + b_2 Q^2(z). \tag{27}$$

Substituting (27) into Eq. (26) and taking into account relations (9) we obtain the system of algebraic equations in the form

$$\begin{gathered}2(c^2-1)a_1+\beta a_1^3-2a_1b_2=0,\\ -3(c^2-1)a_1+3\beta a_0a_1^2-2a_0b_2-2a_1b_1=0,\\ (c^2-1)a_1+3\beta a_0^2a_1+(\omega^2(c^2-1)-\alpha)a_1-2a_0b_1-2a_1b_0=0,\\ (\omega^2(c^2-1)-\alpha)a_0+\beta a_0^3-2a_0b_0=0,\\ -6\beta a_1^2+6(c^2+1)b_2=0,\\ (c^2+1)(2b_1-10b_2)+10\beta a_1^2-4\beta a_0a_1=0,\\ (c^2+1)(4b_2-3b_1)-4\beta a_1^2+6\beta a_0a_1=0,\\ (c^2+1)b_1-2\beta a_0a_1=0.\end{gathered} \tag{28}$$

From (28) we have following values of coefficients $a_0,\ \ a_1,\ \ b_0,\ \ b_1,\ \ b_2$

$$a_0=\pm\frac{c^2+1}{\sqrt{-2\beta(c^2+1)}},\quad a_1=\pm\frac{\sqrt{-2\beta(c^2+1)}}{\beta},\quad b_0=\frac{1}{2}\omega^2(c^2-1)-\frac{\alpha}{2}-\frac{1}{4}(c^2+1), \tag{29}$$

$b_1=2,\ \ b_2=-2.$
Using values of parameters (29) we have

$$y(z)=\pm\sqrt{-\frac{1+c^2}{2\beta}}(1-2Q(z)), \tag{30}$$

$$h(z)=\frac{1}{2}\omega^2(c^2-1)-\frac{\alpha}{2}-\frac{1}{4}(c^2+1)+2Q(z)-2Q^2(z).$$

Combining (30) with (8), we obtain the exact solution to Eq. (26) and the exact solution to coupled Higgs field equation can be written as

$$u(x,t)=\pm\sqrt{-\frac{1+c^2}{2\beta}}e^{i(\omega cx+\omega t)}(1-\frac{2}{1+e^z}), \tag{31}$$

$$v(x,t)=\frac{1}{2}\omega^2(c^2-1)-\frac{\alpha}{2}-\frac{1}{4}(c^2+1)+2(\frac{1}{1+e^z}-\frac{1}{(1+e^z)^2}),$$

where $z=x+ct$.

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