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ON A SUBCLASS OF HARMONIC MAPPINGS

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Dedicated to the retirement of Professor Shigeyoshi Owa

Abstract. In the present paper we extent the fundamental property that if $h(z)$ and $g(z)$ are regular functions in the open unit disc $\mathbb{D}$ with the properties $h(0) = g(0) = 0$, $h(z)$ maps $\mathbb{D}$ onto λ-spiral region and $\text{Re}\left\{e^{i\lambda}\frac{g'(z)}{h'(z)}\right\} > 0$, then $\text{Re}\left\{e^{i\lambda}\frac{g(z)}{h(z)}\right\} > 0$, and then give some applications of this to the harmonic functions.

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1. Introduction

A planar harmonic mapping in the unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ is a complex-valued harmonic function f which maps $\mathbb{D}$ onto some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is simply connected, the mapping f has a canonical decomposition $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$, as usual, we call h the analytic part of f and g the co-analytic part of f. An elegant and complete account of the theory of planar harmonic mapping is given in Duren's monograph [2].

Lewy [4] proved in 1936 that the harmonic function f is locally univalent in a simply connected domain $\mathcal{D}_1$ if and only if its Jacobien

$$J_f(z) = \left|h'(z)\right|^2 - \left|g'(z)\right|^2 > 0$$

is different from zero in $\mathcal{D}_1$. In view of this result, locally univalent harmonic mappings in the unit disc are either sense-reversing if

$$\left|g'(z)\right| > \left|h'(z)\right|$$

in $\mathcal{D}_1$ or sense-preserving if

$$\left|g'(z)\right| < \left|h'(z)\right|$$

in $\mathcal{D}_1$. Throughout this paper we will restrict ourselves to the study of sense-preserving harmonic mappings. However, since f is sense-preserving if and only if $\overline{f}$

is sense-reserving, all the results obtained in this article regarding sense-preserving harmonic mappings can be adapted to sense-reversing ones. Note that $f = h + \overline{g}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ does not vanish in the unit disc and the second-complex dilatation $w(z) = \frac{g'(z)}{h'(z)}$ has the property $|w(z)| < 1$ in $\mathbb{D}$. Therefore we can take $h(z) = z + a_2z^2 + \cdots$, $g(z) = b_1z + b_2z^2 + \cdots$. Thus the class of all harmonic mappings being sense-preserving in the unit disc can be defined by

$$\mathcal{S_H} = \Big\{ f = h(z) + \overline{g(z)} \,|\, h(z) = z + a_2z^2 + \cdots, \\ g(z) = b_1z + b_2z^2 + \cdots, f \text{ sense-preserving} \Big\}.$$

Thus $\mathcal{S_H}$ contains the standard class $\mathcal{S}$ of analytic univalent functions.

Let Ω be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Denote by $\mathcal{P}$, the family of functions $p(z) = 1 + p_1z + p_2z^2 + \cdots$ which are regular in $\mathbb{D}$ such that

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)}$$

for some function $\phi(z) \in \Omega$ for all $z \in \mathbb{D}$.

Next, let $\mathcal{S}^*(\lambda)$ denote the family of functions $s(z) = z + c_2z^2 + c_3z^3 + \cdots$ which are regular in $\mathbb{D}$ such that

$$e^{i\lambda} z \frac{s'(z)}{s(z)} = (\cos\lambda)p(z) + i\sin\lambda \quad \left(|\lambda| < \frac{\pi}{2}\right)$$

for some $p(z) \in \mathcal{P}$ for all $z \in \mathbb{D}$.

Let $s_1(z) = z + \alpha_2z^2 + \alpha_3z_3 + \cdots$ and $s_2(z) = z + \beta_2z^2 + \beta_3z^3 + \cdots$ be analytic functions in $\mathbb{D}$. If there exists $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ for all $z \in \mathbb{D}$. Then we say that $s_1(z)$ is subordinate to $s_2(z)$ and we write $s_1(z) \prec s_2(z)$, then $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$.

Now, we consider the following class of harmonic mappings in the plane:

$$\mathcal{S}^*_{\mathcal{HS}}(\lambda) = \Big\{ f = h(z) + \overline{g(z)} \,|\, h(z) \in \mathcal{S}^*(\lambda), \\ \mathrm{Re}(e^{i\lambda}w(z)) = \mathrm{Re}\left(e^{i\lambda}\frac{g'(z)}{h'(z)}\right) > 0 \Big\}.$$

In the present paper we investigate the class $\mathcal{S}^*_{\mathcal{HS}}(\lambda)$.

2. Main Results

Lemma 1. *Let $h(z)$ be an element of $\mathcal{S}^*(\lambda)$, then*

$$r\mathfrak{A}(\lambda, -r) \leq |h(z)| \leq r\mathfrak{A}(\lambda, r), \quad |z| = r < 1, |\lambda| < \pi/2 \tag{1}$$

where

$$\mathfrak{A}(\lambda, r) = \frac{(1+r)^{\cos\lambda(1-\cos\lambda)}}{(1-r)^{\cos\lambda(1+\cos\lambda)}}.$$

This inequality is sharp because the extremal function is

$$h_*(z) = \frac{z}{(1-z)^{2(\cos\lambda)e^{-i\lambda}}}. \tag{2}$$

Proof. Since $h(z) \in \mathcal{S}^*(\lambda)$, then

$$e^{i\lambda} z \frac{h'(z)}{h(z)} = (\cos\lambda)p(z) + i\sin\lambda \quad \left(|\lambda| < \frac{\pi}{2}, z \in \mathbb{D}\right).$$

Thus, we have

$$e^{i\lambda} z \frac{h'(z)}{h(z)} = (\cos\lambda)\frac{1+\phi(z)}{1-\phi(z)} + i\sin\lambda$$

or

$$z\frac{h'(z)}{h(z)} \prec \frac{1+e^{-2i\lambda}z}{1-z}. \tag{3}$$

Geometrically, the meaning of the relation (3) is that the image of $\mathbb{D}$ lies inside the open disc with the center $C(r) = \left(\frac{1+(\cos 2\lambda)r^2}{1-r^2}, -\frac{\sin 2\lambda}{1-r^2}\right)$ and the radius $\rho(r) = \frac{2(\cos\lambda)r}{1-r^2}$. Therefore we have

$$\left| z\frac{h'(z)}{h(z)} - \frac{1+e^{-2i\lambda}r^2}{1-r^2} \right| \leq \frac{2(\cos\lambda)r}{1-r^2}$$

which gives

$$\frac{1-2(\cos\lambda)r+(\cos 2\lambda)r^2}{r(1-r^2)} \leq \frac{\partial}{\partial r}\log|h(z)| \leq \frac{1+2(\cos\lambda)r+(\cos 2\lambda)r^2}{r(1-r^2)}, \tag{4}$$

integrating the last inequality (4) from 0 to r we obtain (1). □

Corollary 2. *If* $h(z) \in \mathcal{S}^*(\lambda)$*, then*

$$\frac{1}{\mathfrak{B}(\lambda, r)} \leq \left| z\frac{h'(z)}{h(z)} \right| \leq \mathfrak{B}(\lambda, r), \quad |\lambda| < \pi/2,\ |z| = r < 1 \tag{5}$$

where

$$\mathfrak{B}(\lambda, r) = \frac{\sqrt{(1-r^2)^2 + 4(\cos^2\lambda)r^2} + 2(\cos\lambda)r}{1-r^2}.$$

This inequality sharp because the extremal function is given by (2)*.*

Corollary 3. *If* $h(z) \in \mathcal{S}^*(\lambda)$*, then*

$$\frac{\mathfrak{A}(\lambda, -r)}{\mathfrak{B}(\lambda, r)} \leq \left| h'(z) \right| \leq \mathfrak{A}(\lambda, r)\mathfrak{B}(\lambda, r), \quad |\lambda| < \pi/2,\ |z| = r < 1 \tag{6}$$

where

$$\mathfrak{A}(\lambda, r) = \frac{(1+r)^{\cos\lambda(1-\cos\lambda)}}{(1-r)^{\cos\lambda(1+\cos\lambda)}}.$$

and

$$\mathfrak{B}(\lambda, r) = \frac{\sqrt{(1-r^2)^2 + 4(\cos^2\lambda)r^2} + 2(\cos\lambda)r}{1-r^2}.$$

This inequality sharp because the extremal function is given by (2)*.*

Corollary 2 and Corollary 3 are simple consequences of Lemma 1.

Theorem 4. *Let* $f = h(z) + \overline{g(z)}$ *be an element of* $\mathcal{S}^*_{\mathcal{HS}}(\lambda)$*, then* $\frac{g(z)}{h(z)} \in \mathcal{P}$ *for all* $z \in \mathbb{D}$*.*

Proof. Since $f = h(z) + \overline{g(z)} \in \mathcal{S}^*_{\mathcal{HS}}(\lambda)$ satisfies the condition

$$\text{Re}\left(e^{i\lambda}\frac{g'(z)}{h'(z)} \right) > 0,$$

we have

$$\frac{\frac{1}{b_1}g'(z)}{h'(z)} = \frac{1+e^{-2i\lambda}\phi(z)}{1-\phi(z)},\ \ \phi \in \Omega$$

or

$$\frac{\frac{1}{b_1}g'(z)}{h'(z)} \prec \frac{1+e^{-2i\lambda}z}{1-z} \tag{7}$$

for all $z \in \mathbb{D}$. Now, we define the function

$$\frac{G(z)}{h(z)} = \frac{\frac{1}{b_1}g(z)}{h(z)} = \frac{1+e^{-2i\lambda}\phi(z)}{1-\phi(z)} \Leftrightarrow \frac{G(z)}{h(z)} \prec \frac{1+e^{-2i\lambda}z}{1-z} \tag{8}$$

for all $z \in \mathbb{D}$. Then $\phi(z)$ is analytic in $\mathbb{D}$ and $\phi(0) = 0$. Taking the logarithmic differentiation in both sides of (8), we have that

$$\frac{G(z)}{h(z)} = \frac{G'(z)}{h'(z)} - \frac{ce^{i\lambda}z\phi'(z)}{(1-\phi(z))^2}\frac{h(z)}{e^{i\lambda}zh'(z)}, \tag{9}$$

where $c = 1 + e^{-2i\lambda}$. On the other hand, since $h(z)$ is λ-spirallike, then we have

$$\frac{h(z)}{e^{i\lambda}zh'(z)} = \frac{1-\phi(z)}{e^{i\lambda} + e^{-i\lambda}\phi(z)}. \tag{10}$$

Considering (7), (8), (9) and (10) together we obtain

$$F(z) = \frac{G(z)}{h(z)} = \frac{1+e^{-2i\lambda}\phi(z)}{1-\phi(z)} - \frac{cz\phi'(z)}{(1-\phi(z))(1+e^{-2i\lambda}\phi(z))}. \tag{11}$$

Now, it is easy to realize that the subordination (8) is equivalent to $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Indeed, assume the contrary: there exists a $z_1 \in \mathbb{D}$ such that $|\phi(z_1)| = 1$. Then by Jack's Lemma [3], $z_1\phi'(z_1) = k\phi(z_1)$ for some real $k \geq 1$. For such z_1, we have

$$\begin{aligned} F(z_1) = \frac{G(z_1)}{h(z_1)} &= \frac{1+e^{-2i\lambda}\phi(z_1)}{1-\phi(z_1)} - \frac{ck\phi(z_1)}{(1-\phi(z_1))(1+e^{-2i\lambda}\phi(z_1))} \\ &= F(\phi(z_1)) \notin F(\mathbb{D}), \end{aligned}$$

because $|\phi(z_1)| = 1$ and $k \geq 1$. But this contradicts $F(z) = \frac{G(z)}{h(z)} \prec \frac{1+e^{-2i\lambda}z}{1-z}$, so the assumption is wrong, i.e, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$. □

Theorem 5. *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}^*_{\mathcal{HS}}(\lambda)$, then*

$$\frac{|b_1|\mathfrak{A}(\lambda,-r)}{\mathfrak{B}^2(\lambda,r)} \leq |g'(z)| \leq |b_1|\mathfrak{A}(\lambda,r)\mathfrak{B}^2(\lambda,r), \tag{12}$$

where

$$\mathfrak{A}(\lambda,r) = \frac{(1+r)^{\cos\lambda(1-\cos\lambda)}}{(1-r)^{\cos\lambda(1+\cos\lambda)}},$$

and

$$\mathfrak{B}(\lambda,r) = \frac{\sqrt{(1-r^2)^2 + 4(\cos^2\lambda)r^2} + 2(\cos\lambda)r}{1-r^2}$$

for all $|z| = r < 1$.

Proof. Since

$$e^{i\lambda}\frac{g'(z)}{h'(z)} = (\cos\lambda)p(z) + i\sin\lambda$$

then we have

$$e^{i\lambda}\frac{g'(z)}{h'(z)} = (\cos\lambda)\frac{1+\phi(z)}{1-\phi(z)} + i\sin\lambda \quad (\phi \in \Omega).$$

Thus

$$\frac{1}{b_1}\frac{g'(z)}{h'(z)} = \frac{1+e^{-2i\lambda}\phi(z)}{1-\phi(z)}$$

or

$$\frac{1}{b_1\cos\lambda}\left(e^{i\lambda}\frac{g'(z)}{h'(z)} - ib_1\sin\lambda\right) = p(z) \tag{13}$$

for all $z \in \mathbb{D}$. On the other hand, since $p(z) \in \mathcal{P}$, we know that

$$\left|p(z) - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2} \quad (|z| = r < 1).$$

Therefore, we have

$$\left|\frac{g'(z)}{h'(z)} - \frac{b_1(1+e^{-2i\lambda}r^2)}{1-r^2}\right| \leq \frac{|b_1|2(\cos\lambda)r}{1-r^2}$$

or

$$\begin{aligned}\frac{|b_1|\left(|1+e^{-2i\lambda}r^2| - 2(\cos\lambda)r\right)}{1-r^2} &\leq \left|\frac{g'(z)}{h'(z)}\right| \\ &\leq \frac{|b_1|\left(|1+e^{-2i\lambda}r^2| + 2(\cos\lambda)r\right)}{1-r^2}.\end{aligned} \tag{14}$$

We note that the inequality (14) can be written in the form

$$|b_1|\frac{|h'(z)|}{\mathfrak{B}(\lambda,r)} \leq |g'(z)| \leq |b_1|\mathfrak{B}(\lambda,r)|h'(z)|. \tag{15}$$

Using Corollary 3 in the inequality (15) we get (12). □

Theorem 6. *If $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}^*_{\mathcal{HS}}(\lambda)$, then*

$$\begin{aligned}&\frac{\mathfrak{A}^2(\lambda,-r)}{\mathfrak{B}^2(\lambda,r)}\frac{(1+|b_1|r)^2 - (|b_1|+r)^2}{(1+|b_1|r)^2} \leq |J_f| \\ &\leq (\mathfrak{A}(\lambda,r)\mathfrak{B}(\lambda,r))^2\frac{(1-|b_1|r)^2 - (|b_1|-r)^2}{(1-|b_1|r)^2}\end{aligned} \tag{16}$$

for all $|z| = r < 1$, and functions $\mathfrak{A}$ and $\mathfrak{B}$ are defined in Corollary 3.

Proof. Since

$$e^{i\lambda}\frac{g'(z)}{h'(z)} = e^{i\lambda}\frac{(b_1z+b_2z^2+\cdots)'}{(z+a_2z^2+\cdots)'} = e^{i\lambda}\frac{b_1+2b_2z+\cdots}{1+2a_2z+\cdots} = \omega(z),$$

then $\omega(0) = e^{i\lambda}b_1 = b$. Thus

$$\left|e^{i\lambda}\frac{g'(z)}{h'(z)}\right| = |\omega(z)| < 1,$$

then the function

$$\varphi(z) = \frac{\omega(z)-\omega(0)}{1-\overline{\omega(0)}\omega(z)}$$

satisfies the conditions of Schwarz lemma, thus we have

$$\omega(z) = e^{i\lambda}\frac{g'(z)}{h'(z)} = \frac{b+\varphi(z)}{1+\bar{b}\varphi(z)} \Leftrightarrow e^{i\lambda}\frac{g'(z)}{h'(z)} \prec \frac{b+z}{1+\bar{b}z}$$

for all $z \in \mathbb{D}$. On the other hand the transformation $W(z) = \frac{b+z}{1+\bar{b}z}$ maps $|z| = r$ onto the disc with the center

$$C(r) = \left(\frac{\alpha_1(1-r^2)}{1-(\alpha_1^2+\alpha_2^2)r^2}, \frac{\alpha_2(1-r^2)}{1-(\alpha_1^2+\alpha_2^2)r^2}\right)$$

and the radius

$$\rho(r) = \frac{(1-(\alpha_1^2+\alpha_2^2))r}{1-(\alpha_1^2+\alpha_2^2)r^2},$$

where $\alpha_1 = \mathrm{Re}b = \mathrm{Re}(e^{i\alpha}b_1)$, $\alpha_2 = \mathrm{Im}b = \mathrm{Im}(e^{i\alpha}b_1)$. Therefore, using the subordination principle, we can write

$$\left|\frac{g'(z)}{h'(z)} - \frac{b_1(1-r^2)}{1-|b_1|^2r^2}\right| \le \frac{(1-|b_1|^2)r}{1-|b_1|^2r^2}. \tag{17}$$

After the straightforward calculations form (17) we obtain the following inequality

$$\begin{aligned}\frac{(1+|b_1|r)^2-(|b_1|+r)^2}{(1+|b_1|r)^2} &\le \left(1-\left|\frac{g'(z)}{h'(z)}\right|^2\right)\\ &\le \frac{(1-|b_1|r)^2-(|b_1|-r)^2}{(1+|b_1|r)^2},\end{aligned} \tag{18}$$

and than we have

$$\begin{aligned}&|h'(z)|\frac{(1+|b_1|r)^2-(|b_1|+r)^2}{(1+|b_1|r)^2} \le J_f = |h'(z)|\left(1-\left|\frac{g'(z)}{h'(z)}\right|^2\right)\\ &\le |h'(z)|\frac{(1-|b_1|r)^2-(|b_1|-r)^2}{(1-|b_1|r)^2}\end{aligned} \tag{19}$$

for all $|z| = r < 1$. Using the Corollary 3 in the inequality (19) we get (16). □

Corollary 7. *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}^*_{\mathcal{HS}}(\lambda)$, then*

$$\begin{aligned}\int_0^r \frac{\mathfrak{A}(\lambda,-\rho)}{\mathfrak{B}(\lambda,\rho)}\frac{(1-\rho)(1-|b_1|)}{1+|b_1|\rho}d\rho \leq |f| \\ \leq \int_0^r \mathfrak{A}(\lambda,\rho)\mathfrak{B}(\lambda,\rho)\frac{(1+\rho)(1+|b_1|)}{1+|b_1|\rho}d\rho\end{aligned} \tag{20}$$

for $|z| = r < 1$, where $\mathfrak{A}$ and $\mathfrak{B}$ are defined in Corollary 3.

Proof. Since

$$\left|\frac{g'(z)}{h'(z)} - \frac{b_1(1-r^2)}{1-|b_1|^2r^2}\right| \leq \frac{(1-|b_1|^2)r}{1-|b_1|^2r^2}$$

then we have

$$\frac{(1-r)(1+|b_1|)}{1-|b_1|r} \leq 1 + \left|\frac{g'(z)}{h'(z)}\right| \leq \frac{(1+r)(1+|b_1|)}{1+|b_1|r} \tag{21}$$

and

$$\frac{(1-r)(1-|b_1|)}{1+|b_1|r} \leq 1 - \left|\frac{g'(z)}{h'(z)}\right| \leq \frac{(1+r)(1-|b_1|)}{1-|b_1|r}. \tag{22}$$

On the other hand since $f = h(z) + \overline{g(z)}$ is a sense-preserving mapping, then

$$(|h'(z)| - |g'(z)|)|dz| \leq |df| \leq (|h'(z)| + |g'(z)|)|dz|. \tag{23}$$

Using (21), (22), (23) and Corollary 3, we get the desired result. □

Theorem 8. *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}^*_{\mathcal{HS}}(\lambda)$, then*

$$\sum_{k=2}^{n} k^2|b_k - b_1a_k|^2 \leq |1-b_1|^2 + \sum_{k=2}^{n} k^2|a_k - b_kb_1|^2. \tag{24}$$

Proof. The proof of this theorem is based on the Clunie method [1]. Since

$$e^{i\lambda}\frac{g'(z)}{h'(z)} \prec \frac{b+z}{1+\bar{b}z} \Leftrightarrow e^{i\lambda}\frac{g'(z)}{h'(z)} = \frac{b+\varphi(z)}{1+\bar{b}\varphi(z)}$$

then we obtain

$$e^{i\lambda}(g'(z) - h'(z)) = (h'(z) - b_1g'(z))\varphi(z). \tag{25}$$

The equality (25) can be written in the form

$$\sum_{k=2}^{n} e^{i\lambda}k(b_k - a_kb_1)z^k + \sum_{k=n+1}^{\infty} d_kz^k = \left[1 - b_1^2 + \sum_{k=2}^{n} k(a_k - b_kb_1)z^k\right]\varphi(z). \tag{26}$$

Since (26) has the form $K(z) = L(z)\varphi(z)$, where $|\varphi(z)| < 1$, it follows that

$$\frac{1}{2\pi}\int_0^{2\pi} |K(re^{i\theta})|^2 d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |L(re^{i\theta})|^2 d\theta \tag{27}$$

for each r $(0 < r < 1)$. Expressing (27) in the terms of the coefficients in (26), we obtain the inequality

$$\sum_{k=2}^{n} k^2|b_k - a_k b_1|^2 r^{2n} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2n} \le |1-b_1|^2 + \sum_{k=2}^{n} k^2 |a_k - b_1 b_k|^2 r^{2n}. \tag{28}$$

In particular (28) implies

$$\sum_{k=2}^{n} k^2|b_k - a_k b_1|^2 r^{2n} \le |1-b_1|^2 + \sum_{k=2}^{n} k^2 |a_k - b_1 b_k|^2 r^{2n}. \tag{29}$$

By letting $r \to 1$ in (29), we conclude that

$$\sum_{k=2}^{n} k^2|b_k - b_1 a_k|^2 \le |1-b_1|^2 + \sum_{k=2}^{n} k^2 |a_k - b_k b_1|^2.$$

□

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