ABOUT SOME CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. Our subject of study in this paper is represented by a class of convex functions with negative coefficients defined by using a modified Sălăgean operator.

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1. Introduction

Let $\mathcal{H}(U)$ to be the set of functions which are regular in the unit disc U,

$$A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$$

and $S = \{f \in A : f \text{ is univalent in } U\}$.

In [7] the subfamily T of S consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \geq 0, j = 2, 3, ...,\ z \in U \tag{1}$$

was introduced.

The purpose of this paper is to define a class of convex functions with negative coefficients and to give some properties of its by using a modified Sălăgean operator.

2. Preliminary results

Let D^n be the Sălăgean differential operator (see [6]) $D^n : A \to A$, $n \in \mathbb{N}$, defined as:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z))$$

Remark 2.1. If $f \in T, f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...,$ $z \in U$ then $D^n f(z) = z - \sum_{j=2}^{\infty} j^n a_j z^j$.

Definition 2.1. [1] *Let* $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 0$ *and* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. *We denote by* D_λ^β *the linear operator defined by*

$$D_\lambda^\beta : A \to A\,,$$

$$D_\lambda^\beta f(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^\beta \, a_j z^j\,.$$

Theorem 2.1. [6] *If* $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...,$ $z \in U$ *then the next assertions are equivalent:*

(i) $\sum_{j=2}^{\infty} j a_j \leq 1$

(ii) $f \in T$

(iii) $f \in T^*$, *where* $T^* = T \bigcap S^*$ *and* S^* *is the well-known class of starlike functions.*

For $\alpha \in [0, 1)$ and $n \in \mathbb{N}$, we denote

$$S_n^c(\alpha) = \left\{ f \in A : Re \frac{D^{n+2} f(z)}{D^{n+1} f(z)} > \alpha, z \in U \right\}$$

the set of n-convex functions of order α.

Definition 2.2 [5] *Let* $I_c : A \to A$ *be the integral operator defined by* $f = I_c(F)$, *where* $c \in (-1, \infty)$, $F \in A$ *and*

$$f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} F(t) dt. \tag{2}$$

We note if $F \in A$ *is a function of the form (1), then*

$$f(z) = I_c F(z) = z - \sum_{j=2}^{\infty} \frac{c+1}{c+j} a_j z^j. \tag{3}$$

We denote by $f * g$ the modified Hadamard product of two functions $f(z)$, $g(z) \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j, (a_j \geq 0,\ j = 2, 3, ...)$ and $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$, $(b_j \geq 0,$ j=2,3,...), is defined by

$$(f * g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j.$$

An analytic function f is set to be subordinate to an analytic function g if $f(z) = g(w(z))$, $z \in U$, for some analytic function w with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$. We denote this subordination by $f \prec g$.

Theorem 2.2. [4] *If* f *and* g *are analytic in* U *with* $f \prec g$, *then for* $\mu > 0$ *and* $z = re^{i\theta}$ $(0 < r < 1)$, *we have*

$$\int_0^{2\pi} |f(z)|^{\mu} d\theta \leqq \int_0^{2\pi} |g(z)|^{\mu} d\theta.$$

Definition 2.3. [2] *We consider the integral operator* $I_{c+\delta} : A \to A$, $0 < u \leq 1$, $1 \leq \delta < \infty$, $0 < c < \infty$, *defined by*

$$f(z) = I_{c+\delta}(F(z)) = (c + \delta) \int_0^1 u^{c+\delta-2} F(uz) du. \tag{4}$$

Remark 2.2. For $F(z) = z + \sum_{j=2}^{\infty} a_j z^j$. From (4) we obtain

$$f(z) = z + \sum_{j=2}^{\infty} \frac{c+\delta}{c+j+\delta-1} a_j z^j.$$

Also, we notice that $0 < \dfrac{c+\delta}{c+j+\delta-1} < 1$, where $0 < c < \infty$, $j \geq 2$, $1 \leq \delta < \infty$.

Remark 2.3. It is easy to prove that for $F(z) \in T$ and $f(z) = I_{c+\delta}(F(z))$, we have $f(z) \in T$, where $I_{c+\delta}$ is the integral operator defined by (4).

3. Main results

Definition 3.1. *Let $f \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...,$ $z \in U$. We say that f is in the class $T^c L_\beta(\alpha)$ if:*

$$Re \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} > \alpha, \quad \alpha \in [0,1), \quad \lambda \geq 0, \quad \beta \geq 0, \quad z \in U.$$

Theorem 3.1. *Let $\alpha \in [0,1)$, $\lambda \geq 0$ and $\beta \geq 0$. The function $f \in T$ of the form (1) is in the class $T^c L_\beta(\alpha)$ iff*

$$\sum_{j=2}^{\infty} [(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)] a_j < 1-\alpha. \tag{5}$$

Proof. Let $f \in T^c L_\beta(\alpha)$, with $\alpha \in [0,1)$, $\lambda \geq 0$ and $\beta \geq 0$. We have

$$Re \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} > \alpha\,.$$

If we take $z \in [0,1)$, $\beta \geq 0$, $\lambda \geq 0$, we have (see Definition 2.1):

$$\frac{1-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+2}a_j z^{j-1}}{1-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1}a_j z^{j-1}} > \alpha. \tag{6}$$

From (6) we obtain:

$$\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)a_j z^{j-1} < 1-\alpha.$$

Letting $z \to 1^-$ along the real axis we have:

$$\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)a_j < 1-\alpha.$$

Conversely, let take $f \in T$ for which the relation (5) hold.

The condition $Re\dfrac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} > \alpha$ is equivalent with

$$\alpha - Re\left(\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)}-1\right) < 1\,. \tag{7}$$

We have

$$\alpha - Re\left(\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)}-1\right) \leq \alpha + \left|\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)}-1\right|$$

$$= \alpha + \left|\frac{\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1}a_j[(j-1)\lambda]z^{j-1}}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_j z^{j-1}}\right|$$

$$\leq \alpha + \frac{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_j|1-j|\lambda|z|^{j-1}}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_j|z|^{j-1}}$$

$$= \alpha + \frac{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_j(j-1)\lambda|z|^{j-1}}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_j|z|^{j-1}}$$

$$< \alpha + \frac{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_j(j-1)\lambda}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_j} < 1.$$

Thus

$$\alpha + \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}a_j[(j-1)\lambda+1-\alpha] < 1,$$

which is the condition (5).

REMARK 3.1. Using the condition (5) it is easy to prove that $T^cL_{\beta+1}(\alpha) \subseteq T^cL_\beta(\alpha)$, where $\beta \geq 0$, $\alpha \in [0,1)$ and $\lambda \geq 0$.

THEOREM 3.2. *If* $f(z) = z - \sum_{j=2}^{\infty} a_j z^j \in T^cL_\beta(\alpha), (a_j \geq 0,\ j = 2,3,...)$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j \in T^cL_\beta(\alpha)$, $(b_j \geq 0,\ j = 2,3,...)$, $\alpha \in [0,1)$, $\lambda \geq 0$, $\beta \geq 0$, *then* $f(z) * g(z) \in T^cL_\beta(\alpha)$.

Proof. We have

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}[(j-1)\lambda+1-\alpha]a_j < 1-\alpha$$

and

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}[(j-1)\lambda+1-\alpha]b_j < 1-\alpha.$$

We know from the definition of the modified convolution product that $f(z)*g(z) = z - \sum\limits_{j=2}^{\infty} a_j b_j z^j$. From $g(z) \in T$, by using Theorem 2.1, we have $\sum\limits_{j=2}^{\infty} j b_j \leq 1$. We notice that $b_j < 1, \quad j = 2, 3, \dots$.

Thus,

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}[(j-1)\lambda+1-\alpha]a_j b_j < \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}[(j-1)\lambda+1-\alpha]a_j$$

$$< 1-\alpha.$$

This means that $f(z)*g(z) \in T^cL_\beta(\alpha), \quad \beta \geq 0, \quad \alpha \in [0,1)$ and $\lambda \geq 0$.

Theorem 3.3. *If* $F(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j \in T^cL_\beta(\alpha)$, *then* $f(z) = I_c F(z) \in T^cL_\beta(\alpha)$, *where* I_c *is the integral operator defined by (2).*

Proof. We have $f(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j$, where $b_j = \dfrac{c+1}{c+j} a_j, \quad c \in (-1, \infty)$, j=2,3,... .

Thus $\quad b_j < a_j, \quad$ j=2,3 ... and using the condition (5) for $F(z)$ we obtain

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}[(j-1)\lambda+1-\alpha]b_j < \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}[(j-1)\lambda+1-\alpha]a_j < 1-\alpha.$$

This completes our proof.

Theorem 3.4. *Let* $F(z)$ *be in the class* $T^cL_\beta(\alpha)$, $F(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$. *Then* $f(z) = I_{c+\delta}(F(z)) \in T^cL_\beta(\alpha)$, *where* $I_{c+\delta}$ *is the integral operator defined by (4).*

Proof. From $F(z) \in T^cL_\beta(\alpha)$ we have (see Theorem 3.1)

$$\sum_{j=2}^{\infty}[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]a_j < 1-\alpha$$

where $\lambda \geq 0$, $\beta \geq 0$, $0 < c < \infty$ and $1 \leq \delta < \infty$. Let $f(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j$, where (see Remark 3.2)

$$b_j = \frac{c+\delta}{c+\delta+j-1} a_j \geq 0 \;\; and \;\; 0 < \frac{c+\delta}{c+\delta+j-1} < 1\,.$$

From Remark 3.3 we obtain $f(z) \in T$. We have

$$[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]b_j < [(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]a_j\,.$$

Thus,

$$\sum_{j=2}^{\infty}[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]b_j \leq \sum_{j=2}^{\infty}[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]a_j$$
$$< 1-\alpha\,.$$

This completes our proof.

THEOREM 3.5. *Let* $f_1(z) = z$ *and*

$$f_j(z) = z - \frac{1-\alpha}{(1+(j-1)\lambda)^{\beta+1}(1-\alpha+(j-1)\lambda)} z^j,\; j = 2, 3, ...$$

Then $f \in T^c L_\beta(\alpha)$ *iff it can be expressed in the form* $f(z) = \sum\limits_{j=1}^{\infty} \lambda_j f_j(z)$, *where* $\lambda_j \geq 0$ *and* $\sum\limits_{j=1}^{\infty} \lambda_j = 1$.

Proof. Let $f(z) = \sum\limits_{j=1}^{\infty} \lambda_j f_j(z)$, $\lambda_j \geq 0$, j=1,2, ... , with $\sum\limits_{j=1}^{\infty} \lambda_j = 1$. We obtain

$$f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z) = \sum_{j=1}^{\infty} \lambda_j \left(z - \frac{1-\alpha}{[1+(j-1)\lambda]^{\beta+1}[1-\alpha+(j-1)\lambda]} z^j \right)$$

$$= z - \sum_{j=2}^{\infty} \lambda_j \frac{1-\alpha}{[1+(j-1)\lambda]^{\beta+1}[1-\alpha+(j-1)\lambda]} z^j.$$

We have

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}[1-\alpha+(j-1)\lambda]\lambda_j\frac{1-\alpha}{[1+(j-1)\lambda]^{\beta+1}[1-\alpha+(j-1)\lambda]}$$

$$=(1-\alpha)\sum_{j=2}^{\infty}\lambda_j=(1-\alpha)(\sum_{j=1}^{\infty}\lambda_j-\lambda_1)<1-\alpha$$

which is the condition (5) for $f(z)=\sum\limits_{j=1}^{\infty}\lambda_j f_j(z)$. Thus $f(z)\in T^cL_\beta(\alpha)$.

Conversely, we suppose that $f(z)\in T^cL_\beta(\alpha), f(z)=z-\sum\limits_{j=2}^{\infty}a_jz^j, a_j\geq 0$ and we take $\lambda_j=\dfrac{[1+(j-1)\lambda]^{\beta+1}[1-\alpha+(j-1)\lambda]}{1-\alpha}a_j\geq 0$, j=2,3, ..., with $\lambda_1=1-\sum\limits_{j=2}^{\infty}\lambda_j$.

Using the condition (5), we obtain

$$\sum_{j=2}^{\infty}\lambda_j=\frac{1}{1-\alpha}\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}[1-\alpha+(j-1)\lambda]a_j<\frac{1}{1-\alpha}(1-\alpha)=1.$$

Then $f(z)=\sum\limits_{j=1}^{\infty}\lambda_jf_j$, where $\lambda_j\geq 0$, j=1,2, ... and $\sum\limits_{j=1}^{\infty}\lambda_j=1$.

This completes our proof.

Corolary 3.1. *The extreme points of $T^cL_\beta(\alpha)$ are $f_1(z)=z$ and*

$$f_j(z)=z-\frac{1-\alpha}{(1+(j-1)\lambda)^{\beta+1}(1-\alpha+(j-1)\lambda)}z^j,\ j=2,3,\ldots\ .$$

Theorem 3.6. *Let $f(z)\in T^cL_\beta(\alpha)$, $\beta\geq 0$, $\lambda\geq 0$, $\alpha\in[0,1)$, $\mu>0$ and $f_j(z)=z-\dfrac{1-\alpha}{[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]}z^j$, j=2,3... . Then for $z=re^{i\theta}$ $(0<r<1)$, we have*

$$\int_0^{2\pi}|f(z)|^\mu d\theta\leq\int_0^{2\pi}|f_j(re^{i\theta})|^\mu d\theta.$$

Proof We have to show that

$$\int_0^{2\pi}\left|1-\sum_{j=2}^{\infty}a_jz^{j-1}\right|^{\mu}d\theta \le \int_0^{2\pi}\left|1-\frac{1-\alpha}{[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]}z^{j-1}\right|^{\mu}d\theta.$$

From Theorem 2.3 we deduce that it is sufficiently to prove that

$$1-\sum_{j=2}^{\infty}a_jz^{j-1}\prec 1-\frac{1-\alpha}{[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]}z^{j-1}.$$

Considering

$$1-\sum_{j=2}^{\infty}a_jz^{j-1}=1-\frac{1-\alpha}{[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]}w(z)^{j-1}$$

we find that

$$\{w(z)\}^{j-1}=\frac{[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]}{1-\alpha}\sum_{j=2}^{\infty}a_jz^{j-1}$$

which readily yields $w(0)=0$.

By using the condition (5), we can write

$$1-\alpha>[(1+\lambda)^{\beta+1}(1+\lambda-\alpha)]a_2+[(1+2\lambda)^{\beta+1}(1+2\lambda-\alpha)]a_3+...$$

$$+[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]a_j+[(1+j\lambda)^{\beta+1}(1+j\lambda-\alpha)]a_{j+1}+...$$

$$\ge\sum_{i=2}^{\infty}[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]a_i$$

$$=[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]\sum_{i=2}^{\infty}a_i\,.$$

Thus

$$\sum_{j=2}^{\infty}a_j<\frac{1-\alpha}{[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]}$$

and

$$|\{w(z)\}|^{j-1}=\left|\frac{[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]}{1-\alpha}\sum_{j=2}^{\infty}a_jz^{j-1}\right|$$

$$\leq \frac{[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]}{1-\alpha}\sum_{j=2}^{\infty} a_j\,|z|^{j-1} < |z| < 1\,.$$

This completes our theorem's proof.

REMARK 3.2. *We notice that in the particulary case, obtained for $\lambda = 1$ and $\beta \in \mathbb{N}$, we find similarly results for the class $T_n^c(\alpha)$ of the n-convex functions of order α with negative coefficients.*

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