INCLUSION AND NEIGHBORHOOD PROPERTIES OF CERTAIN SUBCLASSES OF ANALYTIC, MULTIVALENT FUNCTIONS OF COMPLEX ORDER INVOLVING CONVOLUTION

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Abstract. In the present investigation, the authors prove several inclusion relations associated with the (n, δ)-neighborhoods of certain subclasses of p-valently analytic functions of complex order, which are introduced here by means of the Hadamard's Convolution. Special cases of some of these inclusion relations are shown to yield many known results.

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1. Introduction, definitions and preliminaries

Let $\mathcal{A}_p(n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (n, p \in \mathbb{N}), \tag{1}$$

which are *analytic* and *p-valent* in the open unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

For a given function $g \in \mathcal{A}_p(n)$ defined by

$$g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k \tag{2}$$

the Hadamard Product (or convolution) $f * g$ of f given by (1) and g given by (5) is defined by

$$(f * g)(z) := z^p + \sum_{k=n+p}^{\infty} a_k b_k z^k \tag{3}$$

We denote by $\mathcal{T}_p(n)$ the subclass of $\mathcal{A}_p(n)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geqq 0 \quad\; n, p \in \mathbb{N}), \tag{4}$$

which are *analytic* and *p-valent* in the open unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

For a given function $g \in \mathcal{A}_p(n)$ defined by

$$g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k \quad (b_k \geqq 0 \quad n, p \in \mathbb{N}), \tag{5}$$

we introduce a new class of functions $\mathcal{R}_g(p, n, b, m, \lambda)$ of functions satisfying the inequality:

$$\left| \frac{1}{b} \left(\frac{z(f * g)^{(m+1)}(z) + \lambda z^2 (f * g)^{(m+2)}(z)}{\lambda z (f * g)^{(m+1)}(z) + (1 - \lambda)(f * g)^{(m)}(z)} - (p - m) \right) \right| < 1 \tag{6}$$

$$(z \in \mathbb{U}; \quad p \in \mathbb{N}; \quad m \in \mathbb{N}_0; \quad 0 \leqq \lambda \leqq 1; \quad b \in \mathbb{C}\backslash\{0\}; \quad p > m).$$

We observe that $\mathcal{R}_g(p, n, b, m, 0) = \mathcal{S}_g(p, n, b, m)$ [7] and if $g(z) = \dfrac{z^p}{1 - z^n}$, then the class $\mathcal{R}_g(p, n, b, m, \lambda)$ reduces to $\mathcal{R}^p_{n,m}(\lambda, b)$ [10].

We note that there are several interesting new or known subclasses of our function class $\mathcal{R}_g(p, n, b, m, \lambda)$. For example, if we set

$$\lambda = 0, \; m = 1 \;\text{ and }\; \mathrm{b} = \mathrm{p}(1 - \alpha) \quad (\mathrm{p} \in \mathbb{N}; 0 \leq \alpha < 1),$$

$\mathcal{R}_g(p,n,b,m,\lambda)$ reduces to the class studied by Ali *et al.*[1].

Next, following the earlier investigations by Goodman [4], Ruscheweyh [9] and others including Altintas *et al.* [3] (see also [2], [6] and [11]), Murugusundaramoorthy and Srivastava [6], Raina and Srivastava [8] (see also [11]), we define the (n,δ)-neighborhood of a function $f \in \mathcal{T}_p(n)$ by (see, for details, [3, p.1668])

$$N_{n,\delta}(f) = \left\{ h \in \mathcal{T}_p(n) : h(z) = z^p - \sum_{k=n+p}^{\infty} c_k z^k \text{ and } \sum_{k=n+p}^{\infty} k|a_k - c_k| \leq \delta \right\}. \tag{7}$$

It follows from (7) that, if

$$e(z) = z^p \quad (p \in \mathbb{N}), \tag{8}$$

then

$$N_{n,\delta}(e) := \left\{ h \in \mathcal{T}_p(n) : h(z) = z^p - \sum_{k=n+p}^{\infty} c_k z^k \text{ and } \sum_{k=n+p}^{\infty} k|c_k| \leq \delta \right\}. \tag{9}$$

Finally, we denote by $\mathcal{L}_g(p,n,b,m,\lambda)$ the subclass of $\mathcal{T}_p(n)$ consisting of functions $f(z)$ which satisfy the inequality

$$\left| \frac{1}{b} \left((f*g)^{(m+1)}(z) + \lambda z (f*g)^{(m+2)}(z) - (p-m) \right) \right| < p - m \tag{10}$$

$$(z \in \mathbb{U}; \quad p \in \mathbb{N}; \quad m \in \mathbb{N}_0; \quad 0 \leqq \lambda \leqq 1; \quad b \in \mathbb{C}\backslash\{0\}; p > m).$$

Our definitions of the function classes $\mathcal{R}_g(p,n,b,m,\lambda)$ and $\mathcal{L}_g(p,n,b,m,\lambda)$ are motivated essentially by the earlier investigation of Orhan and Kamali [5], in which further details and closely-related subclasses can be found.

The main object of the present paper is to investigate the various properties and characteristics of analytic p-valent functions belonging to the subclasses

$$\mathcal{R}_g(p,n,b,m,\lambda) \quad \text{and} \quad \mathcal{L}_g(p,n,b,m,\lambda)$$

which we have defined here. Apart from deriving a set of coefficient bounds and coefficient inequalities for each of these function classes, we establish several inclusion relationships involving the (n,δ)-neighborhoods of analytic p-valent functions (with negative and missing coefficients) belonging to these subclasses.

2. Coefficient bounds and coefficient inequalities

In this section, we prove the following characterization for functions to be in the subclasses $\mathcal{R}_g(p,n,b,m,\lambda)$ and $\mathcal{L}_g(p,n,b,m,\lambda)$.

Theorem 1. *Let $f \in \mathcal{T}_p(n)$ be given by (4). Then $f \in \mathcal{R}_g(p,n,b,m,\lambda)$ if and only if*

$$\sum_{k=n+p}^{\infty} \binom{k}{m}(1+\lambda(k-m-1))(k-p+|b|)a_k b_k \leq |b| \left(\binom{p}{m}(1+\lambda(p-m-1)) \right), \tag{11}$$

where

$$\binom{k}{m} = \frac{k(k-1)\dots(k-m+1)}{m!}.$$

Proof. Assume that $f \in \mathcal{R}_g(p,n,b,m,\lambda)$. Then, in view of (5) and (1), we have the following inequality:

$$\Re\left(\frac{z(f*g)^{(m+1)}(z) + \lambda z^2 (f*g)^{(m+2)}(z)}{\lambda z (f*g)^{(m+1)}(z) + (1-\lambda)(f*g)^{(m)}(z)} - (p-m) \right) > -|b| \tag{12}$$

which gives

$$\Re\left(\frac{\sum_{k=n+p}^{\infty} \binom{k}{m}(1+\lambda(k-m-1))(p-k)a_k b_k z^{k-m}}{\binom{p}{m}(1+\lambda(p-m-1)z^{p-m} - \sum_{k=n+p}^{\infty} \binom{k}{m}(1+\lambda(k-m-1))a_k b_k z^{k-m}} \right) > -|b|. \tag{13}$$

Setting $z = r\,(0 \leqq r < 1)$ in (13), we observe that the expression in the denominator on the left hand side of (13) is positive for $r = 0$ and also for all $r\,(0 < r < 1)$. Thus by letting $r \to 1^-$ through real values, (13) leads us to the desired assertion (11) of Theorem 1.

Conversely, by applying (11) and setting $|z| = 1$, we find by using (4) that

$$\left| \frac{z(f*g)^{(m+1)}(z) + \lambda z^2 (f*g)^{(m+2)}(z)}{\lambda z (f*g)^{(m+1)}(z) + (1-\lambda)(f*g)^{(m)}(z)} - (p-m) \right|$$

$$= \left| \frac{\sum_{k=n+p}^{\infty} \binom{k}{m}(1+\lambda(k-m-1))(p-k)a_k b_k z^{k-m}}{\binom{k}{m}(1+\lambda(p-m-1))z^{p-m} - \sum_{k=n+p}^{\infty} \binom{k}{m}(1+\lambda(k-m-1))a_k b_k z^{k-m}} \right|$$

$$\leq \frac{|b| \left[\binom{p}{m}(1+\lambda(p-m-1)) - \sum_{k=n+p}^{\infty} \binom{k}{m}(1+\lambda(k-m-1))a_k b_k \right]}{\binom{p}{m}(1+\lambda(p-m-1)) - \sum_{k=n+p}^{\infty} \binom{k}{m}(1+\lambda(k-m-1))a_k b_k}$$

$= |b|$.

Hence, by the maximum modulus principle, we infer that $f \in \mathcal{R}_g(p, n, b, m, \lambda)$, which evidently completes the proof of Theorem 1.

Remark 1. For the choices of $\lambda = 0$ and

$$g(z) = z^p + \sum_{k=n+p}^{\infty} \binom{\mu+k-1}{k-p} z^k \qquad (\mu > -p)$$

Theorem 1 corresponds to a recent result of Raina and Srivastava [8].

Remark 2. For the choice of

$$g(z) := \frac{z^p}{1-z^n} \tag{14}$$

Theorem 1 corresponds to a recent result of Srivastava and Orhan [10].

Remark 3. In the special case when

$$m = 0, \quad p = 1, \quad b = \beta\gamma \quad (0 < \beta \leq 1; \quad \gamma \in \mathbb{C} \backslash \{0\}), \tag{15}$$

and

$$g(z) := \frac{z^p}{1-z^n} \tag{16}$$

Theorem 1 corresponds to a result given earlier by Altintaş [2, p.64, Lemma 1]

By using the same arguments as in the proof of Theorem 1, we can establish Theorem 2 below.

Theorem 2. *Let $f \in \mathcal{T}_p(n)$ be given by (4). Then $f \in \mathcal{L}_g(p,n,b,m,\lambda)$ if and only if*

$$\sum_{k=n+p}^{\infty} \binom{k}{m}(k-m)(1+\lambda(k-m-1))a_k b_k \leqq$$

$$\leqq (p-m)\left[\frac{|b|-1}{m!} + \binom{p}{m}(1+\lambda(p-m-1))\right]. \tag{17}$$

Remark 4. Making use of the same choice of g as mentioned in the Remark 2, Theorem 2 yields the following known result due to Srivastava and Orhan [10].

Theorem 3. *Let $f \in \mathcal{T}_p(n)$ be given by (4). Then $f \in \mathcal{L}(p,n,b,m,\lambda)$ if and only if*

$$\sum_{k=n+p}^{\infty} \binom{k}{m}(k-m)(1+\lambda(k-m-1))a_k \leqq$$

$$\leqq (p-m)\left[\frac{|b|-1}{m!} + \binom{p}{m}(1+\lambda(p-m-1))\right]. \tag{18}$$

3. Inclusion relationships involving (n,δ)-neighborhoods

In this section, we establish several inclusion relationships for the function classes $\mathcal{R}_g(p,n,b,m,\lambda)$ and $\mathcal{L}_g(p,n,b,m,\lambda)$ involving the (n,δ)-neighborhood defined by (9).

Theorem 4. *If $b_k \geqq b_{n+p}$ $(k \geqq n+p)$ and*

$$\delta := \frac{(n+p)|b|\binom{p}{m}(1+\lambda(p-m-1))}{(n+|b|)\binom{n+p}{m}(1+\lambda(n+p-m-1))b_{n+p}} \qquad (p > |b|), \tag{19}$$

then

$$\mathcal{R}_g(p,n,b,m,\lambda) \subset N_{n,\delta}(e). \tag{20}$$

Proof Let $f \in \mathcal{R}_g(p,n,b,m,\lambda)$. Then in view of the assertion (11) of Theorem 1, we have

$$(n+|b|)\binom{n+p}{m}(1+\lambda(n+p-m-1))b_{n+p}\sum_{k=n+p} a_k \leq |b|\binom{p}{m}(1+\lambda(p-m-1)).$$

This yields

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{|b|\binom{p}{m}(1+\lambda(p-m-1))}{(n+|b|)\binom{n+p}{m}(1+\lambda(n+p-m-1))b_{n+p}}. \tag{21}$$

Applying the assertion (11) of Theorem 1 again, in conjunction with (21), we obtain

$$\begin{aligned}
&\binom{n+p}{m}(1+\lambda(n+p-m-1))\sum_{k=n+p} ka_k \\
&\leq |b|\binom{p}{m}(1+\lambda(p-m-1)) + (p-|b|)\binom{n+p}{m}(1+\lambda(n+p-m-1))b_{n+p}\sum_{k=n+p} a_k \\
&\leq |b|\binom{p}{m}(1+\lambda(p-m-1)) + (p-|b|)\binom{n+p}{m}(1+\lambda(n+p-m-1))b_{n+p} \\
&\qquad \cdot \frac{|b|\binom{p}{m}(1+\lambda(p-m-1))}{(n+|b|)\binom{n+p}{m}(1+\lambda(n+p-m-1))b_{n+p}} \\
&= |b|\binom{p}{m}(1+\lambda(p-m-1))\left(\frac{n+p}{n+|b|}\right).
\end{aligned}$$

Hence

$$\sum_{k=n+p} ka_k \leq \frac{|b|(n+p)\binom{p}{m}(1+\lambda(p-m-1))}{(n+|b|)\binom{n+p}{m}(1+\lambda(n+p-m-1))b_{n+p}} := \delta \quad (p > |b|), \tag{22}$$

which, by virtue of (9), establishes the inclusion relation (20) of Theorem 4.

Analogously, by applying the assertion (18) of Theorem 2 instead of the assertion (11) of Theorem 1 to functions in the class $\mathcal{L}_g(p,n,b,m,\lambda)$, we can prove the following inclusion relationship.

Theorem 5. *If* $b_k \geqq b_{n+p}$ $(k \geqq n+p)$ *and*

$$\delta = \frac{(p-m)\left[\frac{|b|-1}{m!} + \binom{p}{m}(1+\lambda(p-m-1))(n+p)\right]}{\binom{n+p}{m}(n+p-m)(1+\lambda(p-m-1)b_{n+p}}, \tag{23}$$

then

$$\mathcal{L}_g(p,n,b,m,\lambda) \subset N_{n,\delta}(e). \tag{24}$$

Remark 5. Applying the parametric substitutions listed in (14), Theorems 4 and 5 would yield the known results due to Srivastava and Orhan [10]. Also, for the special choices mentioned in (14) and (15), Theorems 4 and 5 at once reduces to the result obtained by Altintas *et al. [2].*

4. Neighborhood properties

In this concluding section, we determine the neighborhood properties for each of the following (slightly modified) function classes:

$$\mathcal{R}_g(p,n,b,m,\lambda) \quad \text{and} \quad \mathcal{L}_g(p,n,b,m,\lambda).$$

Here the class $\mathcal{R}_g(p,n,b,m,\lambda)$ consists of functions $f \in \mathcal{T}_p(n)$ for which there exists another function $h \in \mathcal{R}_g(p,n,b,m,\lambda)$ such that

$$\left|\frac{f(z)}{h(z)} - 1\right| < p - \alpha \quad (z \in \mathbb{U}; \quad 0 \leqq \alpha < p) \tag{25}$$

Analogously, the class $\mathcal{R}_g(p,n,b,m,\lambda)$ consists of functions $f \in \mathcal{T}_p(n)$ for which there exists another function $h \in \mathcal{L}_g(p,n,b,m,\lambda)$ satisfying the inequality (25).

Theorem 6. *Let* $h \in \mathcal{R}_g(p,n,b,m,\lambda)$. *Suppose also that*

$$\alpha = p - \frac{\delta}{n+p}$$

$$\cdot \left[\frac{(n+|b|)\binom{n+p}{m}(1+\lambda(n+p-m-1))b_{n+p}}{(n+|b|)\binom{n+p}{m}(1+\lambda(n+p-m-1))b_{n+p} - |b|\binom{p}{m}(1+\lambda(p-m-1))} \right]. \tag{26}$$

Then

$$N_{n,\delta}(h) \subset \mathcal{R}_g(p,n,b,m,\lambda). \tag{27}$$

Proof. Suppose that $f \in N_{n,\delta}(h)$. We then find from (7) that

$$\sum_{k=n+p}^{\infty} k|a_k - c_k| \leqq \delta \tag{28}$$

which readily implies the coefficient inequality

$$\sum_{k=n+p}^{\infty} |a_k - c_k| \leqq \frac{\delta}{n+p} \qquad (n \in \mathbb{N}). \tag{29}$$

Next, since $h \in \mathcal{R}_g(p,n,b,m,\lambda)$, we have

$$\sum_{k=n+p}^{\infty} c_k \leqq \frac{|b|\binom{p}{m}(1+\lambda(p-m-1))}{(n+|b|)\binom{n+p}{m}(1+\lambda(n+p-m-1))b_{n+p}}, \tag{30}$$

so that

$$\left|\frac{f(z)}{h(z)} - 1\right| < \frac{\sum_{k=n+p}^{\infty} |a_k - c_k|}{1 - \sum_{k=n+p}^{\infty} c_k}$$

$$
\begin{aligned}
&\leq \frac{\delta}{n+p}\left(\frac{1}{1-\dfrac{|b|\binom{p}{m}(1+\lambda(p-m-1))}{(n+|b|)\binom{n+p}{m}(1+\lambda(n+p-m-1))b_{n+p}}}\right)\\
&\leq \frac{\delta}{n+p}\left(\frac{(n+|b|)\binom{n+p}{m}(1+\lambda(n+p-m-1))b_{n+p}}{(n+|b|)\binom{n+p}{m}(1+\lambda(n+p-m-1))b_{n+p}-|b|\binom{p}{m}(1+\lambda(p-m-1))}\right)\\
&= p-\alpha,
\end{aligned}
$$

provided that α is given precisely by (26). Thus, by definition, $f \in \mathcal{R}_g(p,n,b,m,\lambda)$ for α given by (26). This evidently completes the proof of Theorem 6.

The proof of Theorem 7 below is much similar of Theorem 6, hence the proof is omitted.

Theorem 7. *Let $h \in \mathcal{L}_g(p,n,b,m,\lambda)$. Suppose also that*

$$
\alpha = p-\frac{\delta}{n+p}\left(\frac{\binom{n+p}{m}(n+p-m)(1+\lambda(n+p-m-1))b_{n+p}}{\binom{n+p}{m}(n+p-m)(1+\lambda(n+p-m-1))b_{n+p}-(p-m)\left(\frac{|b|-1}{m!}+\binom{p}{m}(1+\lambda(p-m-1))\right)}\right). \tag{31}
$$

Then

$$
N_{n,\delta}(h) \subset \mathcal{L}_g(p,n,b,m,\lambda). \tag{32}
$$

Remark 6. Applying the parametric substitutions listed in (14), Theorems 6 and 7 would yield the known results due to Srivastava and Orhan [10]. Also, for the special choices mentioned in (14) and (15), Theorems 4 and 5 at once reduces to the result obtained by Altintas et al. [2].

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