MAXIMIZING AND MINIMIZING FUZZY SETS

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Abstract. In the problems of A. I., we can distinguish between *Classification, Searching and Representation Methods.* Searching Procedures include the *Blind Search*, without previous knowledge of the domain, and the *Heuristic Search*, with knowledge of such domain.

The *Theory of Fuzzy Functions* provide us with very useful tools for the representation and solution of such problems. Many techniques of classical mathematics are very useful in this area. But we also need some new tools. And they are necessary in essential questions such as the analysis of crisp/fuzzy functions in fuzzy/crisp domains, for instance.

In this search we need to introduce two new fuzzy sets: the *Maximizing Fuzzy Set, M*, and the *minimizing Fuzzy Set, m,* associated to the function (crisp or fuzzy), *f*. We define both sets by μ_{M_f} and μ_{m_f},reflecting the possibility of reaching the maximum/minimum value in each point of the universal set where f is defined. To obtain such values we use their degrees of membership. Furthermore, we show their application by some clear and detailed examples.

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1. Introduction

In the Search Process, we associate *states* with *nodes* and *arcs* or links with *operators*. The process consists of a progressive description: how, departing from the initial node, and selecting in each step the most plausible link, we can reach the final node.

We distinguish between:

- the *Blind Search* (without information of domain, which obliges, therefore, to an exhaustive exploration of the nodes, in the case of a graph).
- the *Heuristic Search* (with information of the domain, which allows the possibility of election between different paths, in the searching tree, because we have at our disposal information about the domain).

In the solution of such problems we need not only the Classical Analysis, but a new tool, the Fuzzy Analysis, which in aspects such as optimization or control consider the treatment of uncertainty.

2. Maximizing fuzzy sets

Let $f : U \to \mathbb{R}$ be a real function defined on the universal set, U.

If we consider its supremum and infimum, $sup(f)$ and $inf(f)$, it is possible to define a new set: M_f, the *maximizing fuzzy set.*

Because taking the range of f, that is, the interval of the real line:

$$R_f = [\inf(f), \ \sup(f)].$$

We will assign to each $x \in U$, a numerical value in the unit interval: $\mu_M(x)$, and so, to introduce the new set:

$$M_f = \{x \mid \mu_M(x)\}_{x \in U}$$

where:

$$\mu_{M_f}(x) = \frac{f(x) - \inf(f)}{\sup(f) - \inf(f)}.$$

Obviously, it shows the proportion between the distance of the function value from the infimum and the width of the range. Therefore, the relative normalized position of $f(x)$ in $R_f \subset \mathbb{R}$.

Graphically, will be:

$\circ \quad \to \quad \circ \quad \to \quad \circ$

inf $\quad f \quad$ sup

$\overleftrightarrow{f(x) \quad - \quad \inf(f)}$

$\overleftrightarrow{\sup(f) \quad - \quad \inf(f)}$

So, we define the new fuzzy set, M_f, by the possibility that x takes the maximum value of f.

3.Minimizing fuzzy sets

In a similar way, we can introduce the *minimizing fuzzy set*,denoted m.By symmetry, we say that it is *the Maximizing fuzzy set of - f:*

$$m_f = m(f) = M(-f) = M_{-f}.$$

Equivalent to define the membership degree of each element of its domain, $D_f \subset U$, as:

$$\mu_{m_f}(x) = \frac{(-f)(x) \quad - \quad \inf(-f)}{\sup(-f) \quad - \quad \inf(-f)}.$$

But according to the known results:

$$\sup(-f) = -\inf(f)\,, \ -\inf(-f) = \sup(f)\,, \ (-f)(x) = -\{f(x)\}\,,$$

we have:

$$\mu_{m_f}(x) = \frac{\sup(f) \quad - \quad f(x)}{\sup(f) \quad - \quad \inf(f)}.$$

As in the precedent case, the sequence of steps will be:

$$x \to \mu_{m_f}(x) \to m_f = \left\{x \mid \mu_{m_f}(x)\right\}_{x \in D_f}.$$

In geometrical interpretation:

$$\begin{array}{ccccc} \circ & \to & \circ & \to & \circ \\ \inf & & f & & \sup \end{array}$$

$$\overleftrightarrow{\sup(f) \quad - \quad f(x)}$$

$$\overleftrightarrow{\sup(f) \qquad - \qquad \inf(f)}$$

Similarly, it shows the proportion between the distance from the function value to the supremum and the width of the range.

As an *example*, we can consider the *sinus* function in the domain $[0,\ 2\pi]$.

In such case, the Maximizing Fuzzy Set, M_f, and the minimizing Fuzzy Set, m_f, are defined by their membership function:

$$\mu_{M_f}(x) = \frac{\sin\ x - \inf\limits_{x\in[0,\ 2\pi]} (\sin x)}{\sup\limits_{x\in[0,\ 2\pi]} (\sin x) - \inf\limits_{x\in[0,\ 2\pi]} (\sin x)}$$

$$= \frac{\sin\ x - (-1)}{1-(-1)} = \frac{1\ +\ \sin\ x}{2} = \frac{1}{2} + \frac{\sin\ x}{2};$$

$$\mu_{m_f}(x) = \frac{\sup\limits_{x\in[0,\ 2\pi]} (\sin\ x) - \sin(x)}{\sup\limits_{x\in[0,\ 2\pi]} (\sin\ x) - \inf\limits_{x\in[0,\ 2\pi]} (\sin\ x)} = \frac{1-\sin\ x}{1-(-1)} = \frac{1}{2} - \frac{\sin\ x}{2}.$$

Where $\mu_{M_f}(x)$ and $\mu_{m_f}(x)$ denote the possibilitity of f taking in x its maximum/minimum value.

So, for instance, if $x = \frac{\pi}{2}$, then:

$$f(x) = \sin\frac{\pi}{2} = 1,$$

with:

$$\mu_{M_f}\left(\frac{\pi}{2}\right) = \frac{1}{2} + \frac{\sin\frac{\pi}{2}}{2} = 1,\ \mu_{m_f}\left(\frac{\pi}{2}\right) = \frac{1}{2} - \frac{\sin\frac{\pi}{2}}{2} = 0.$$

Therefore, the possibility of f taking its maximum value in $\frac{\pi}{2}$ is maximal, and its minimum is minimal.

Also there exist points in the domain with values of the membership degrees, $\mu_{M_f}(x)$ *and* $\mu_{m_f}(x)$, in the interior of the unit interval.

So, if we take: $x = \pi$, then:

$$\mu_{M_f}(\pi) = \frac{1}{2} + \frac{\sin\pi}{2} = \frac{1}{2},\ \mu_{m_f}(\pi) = \frac{1}{2} - \frac{\sin\pi}{2} = \frac{1}{2}.$$

The possibilities of reaching the maximum and the minimum of f in $x = \pi$ are the same, 0.5, and precisely, then, both with the greatest uncertainty. This reflects their maximal distance to both situations.

4. Maximum value

We must distinguish between two cases, according to the presence of a *crisp domain* or a *fuzzy domain, D,* for the function f.

First case:
Suppose that such function reaches its maximum in some point, $x_0 \in D_f$. Then, we can use the Maximizing set of f, M_f, in the search of such point, because it must be the element in D_f that enables to reach the maximum value for the membership function associated to the M_f set:

$$\mu_{M_f}(x_0) = \max_{x \in D_f} \left\{ \mu_{M_f}(x) \right\}.$$

Also it can be written in this way:

$$\mu_{M_f}(x_0) = \max_{x \in D_f} \left[\min \left\{ \mu_{M_f}(x),\ \mu_{D_f}(x) \right\} \right].$$

Second case:
Now, we suppose the function f, defined on the *fuzzy* domain D_f. And we also suppose that the function reaches its maximum in the point $x_0 \in D_f$.Then, we establish this condition:

$$\mu_{M_f}(x) \qquad \text{and} \qquad \mu_{D_f}(x)$$

where both must be considered in its max values.

If we take any element $x_\# \in D_f$, the possibility of reaching in it the maximum value of f is determined by the minimum value between:

$$\mu_{M_f}(x_\#) \qquad \text{and} \qquad \mu_{D_f}(x_\#).$$

In conclusion, the point (x_0) in which the maximum of f is reached, determined by:

$$\mu(x_0) = \max \left[\min \left\{ \mu_{M_f}(x),\ \mu_{D_f}(x) \right\} \right].$$

The maximum value is obviously $f(x_0)$.

5.Minimum value

In a similar way, we can define the minimum value, also into the first case, swaping the roles of:

$$\text{maximizing fuzzy set} \quad \leftrightarrow \quad \text{minimizing fuzzy set.}$$

That is, simbolically, $M_f \leftrightarrow m_f$ or equivalently $\mu_{M_f} \leftrightarrow \mu_{m_f}$.

Therefore, f reaches its maximum value, $f(x_{0^*})$, in the point $x_{0^*} \in D_f$,being:

$$\mu_{m_f}(x_0) = \max_{x \in D_f} \left[\min \left\{ \mu_{m_f}(x),\ \mu_{D_f}(x) \right\} \right].$$

Such as in the precedent situation, it suffices with the change:

$$M_f \leftrightarrow m_f.$$

As easy example, we can give the function:

$$f(x) = 2 - x$$

with:

$$D_f = [0, 1].$$

We have also:

$$\mu_{D_f}(x) = x^2$$

and the Maximizing fuzzy set, M_f, described by:

$$\mu_{M_f}(x) = \frac{f(x) - \inf(f)}{\sup(f) - \inf(f)} = \frac{(2-x) - \inf_{x \in [0,1]} (2-x)}{\sup_{x \in [0,1]} (2-x) - \inf_{x \in [0,1]} (2-x)}$$

$$= \frac{(2-x) - 1}{2 - 1} = 1 - x.$$

Then, because:

$$\mu_f(x_0) = \max\left[\min\left\{\mu_{M_f}(x),\ \mu_{D_f}(x)\right\}\right]$$

we conclude that the point x_0 is reached when both values coincides:

$$\mu_{M_f}(x) = \mu_{D_f}(x).$$

And this implies, according, $0 \leq x \leq 1$, and identifying both membership degrees in x_0, $1 - x_0 = x_0^2$.

So, f obtains its maximum value in, $x_0 \cong 0.6$, where take the value $f(0.6) = 1.4$.

Another different and illustrative *example* can be:

Let's find the maximum value of the crisp function, $f(x) = \cos x$ into the fuzzy domain $D_f = [0,\ 2\pi]$.

We give as membership degree of its elements:

$$\mu_{D_f}(x) = \min\left\{1,\ \frac{x}{\pi}\right\}.$$

Then, we find easily that:

$$\mu_{M_f}(x) = \frac{\cos\ x - \inf\limits_{x\in[0,\ 2\pi]}(\cos x)}{\sup\limits_{x\in[0,\ 2\pi]}(\cos x) - \inf\limits_{x\in[0,\ 2\pi]}(\cos x)}$$

$$= \frac{\cos\ x - (-1)}{1-(-1)} = \frac{1\ +\ \cos\ x}{2} = \frac{1}{2} + \frac{\cos\ x}{2};$$

$$\mu_{m_f}(x) = \frac{\sup\limits_{x\in[0,\ 2\pi]}(\cos\ x)\ -\ \cos(x)}{\sup\limits_{x\in[0,\ 2\pi]}(\cos\ x)\ -\ \inf\limits_{x\in[0,\ 2\pi]}(\cos\ x)}$$

$$= \frac{1-\cos\ x}{1-(-1)} = \frac{1}{2} - \frac{\cos\ x}{2}.$$

So, the maximum (minimum) value is fully reached in $x_0(x_0^*)$, when:

$$\mu_{M_f}(x_0) = \max\left[\min\left\{\mu_{M_f}(x),\ \mu_{D_f}(x)\right\}\right] \Rightarrow x_0 = 0,\ 2\pi;$$

$$\mu_{m_f}\left(x_0^*\right) = \max\left[\min\left\{\mu_{m_f}\left(x\right),\ \mu_{D_f}\left(x\right)\right\}\right] \Rightarrow x_0^* = \frac{\pi}{2}, \frac{3\pi}{2}.$$

6. Conclusion

I hope to contribute with these remarks to the study of fuzzy extrema.

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