# ON CHOOSING PROPER LINGUISTIC DESCRIPTION FOR FRACTIONAL FUNCTIONS IN FUZZY OPTIMIZATION

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Abstract. When a fractional criterion is to be described by linguistic variables two kinds of approaches are possible: to treat it as a unity (nominator/denominator) and to construct a classic membership function or to aggregate the membership functions separately associated to nominator and denominator respectively. In the paper, two propositions are formulated related to choosing possibilities of aggregation coefficients for fractional criteria' membership functions and also mathematical model of multiple linear fractional objective programming is analyzed with respect to linguistic variables based solving methods. Computational results are developed in order to highlight theoretical remarks related to membership functions' for efficiency needed properties.

*Keywords:* fuzzy optimization, linguistic variable, multiple objective programming, linear fractional programming.

## 1. Introduction

The concept of linguistic variable was introduced by Zadeh [10] to provide a means of approximate characterization of phenomena that are too complex or too vague (not well)-defined to be described in conventional quantitative terms. Each linguistic variable involves finite collection of primary terms. Different modifiers (very, more or less, fairly, slightly) could be used with syntactic rules to build well formed sentences. Well formed sentences are combination of modifiers and primary terms. Semantics describe how the membership function of a well formed sentence is calculated. In the case of optimization classic linguistic terms are "close", "quite close", "non very close" and so on in order to give information about the current value of functions in comparison with the expected values of them.

When a fractional criterion is to be described by linguistic variables two kinds of approaches are possible: to treat it as a unity (nominator/denominator) and to construct a classic membership function or to aggregate the membership functions separately associated to nominator and denominator respectively.

In 2004 Rommelfanger ([7]) presented the advantages of fuzzy models in practical use. He points out that some interactive fuzzy solution algorithms provide the opportunity to solve mixed integer programming models as well. In 2002, Liu [4] presented a brief review on fuzzy programming models and classified them into three classes: expected value models, chance-constraint programming and dependent-chance programming. A general method to solve fuzzy programming models was also documented in Liu's paper.

Dutta, Tiwari and Rao [3] modified Luhandjula's linguistic approach ([5]) to obtain efficient solutions for MOLFPP (multiple objective linear fractional programming). In [9] some shortcomings are pointed out and a correct proof of Dutta's et al. main theorem is given. Moreover, it is noticed that the method presented in [3] only works efficiently if some quite restrictive hypotheses are satisfied. Chakraborty and Gupta [2] described a new fuzzy method to solving MOLFPP improving the complexity of computations by defining fuzzy goals for a deterministic MOLFPP.

In [1] Carlsson et al. considered a mathematical programming problem in which the functional relationship between the decision variables and the objective function is not completely known and built a knowledge-base which consists of a block of fuzzy if-then rules, where the antecedent part of the rules contains some linguistic values of the decision variables, and the consequence part is a linear combination of the crisp values of the decision variables.

Mathematical model of multiple criteria linear fractional programming problem is presented in Section 2. Our discussion is limited and exemplified by linear fractional programming problem but it could be developed for a larger class of fractional programming problems. Two propositions related to choosing possibilities of aggregation coefficients for fractional criteria' membership functions are formulated in Section 3. Some computational results are developed in Section 4 in order to highlight theoretical remarks which were made in previous section. Brief summary and conclusions are inserted in Section 5. This paper was presented, in part, in [6].

## 2. Mathematical model of MOLFPP

The mathematical model of an optimization problem is used here as a start point. Membership functions are used to model objectives. Generally, the usefulness of their properties is discussed in connection with optimization problems.

Consider the multiple objective linear fractional programming (MOLFPP):

$$\text{"}\max\text{"}\left\{ z\left(x\right) = \left( \frac{N_1\left(x\right)}{D_1\left(x\right)}, \frac{N_2\left(x\right)}{D_2\left(x\right)}, ..., \frac{N_p\left(x\right)}{D_p\left(x\right)} \right) \mid x \in X \right\} \tag{1}$$

where

(i) $X = \{x \in R^n \mid Ax \leq b, x \geq 0\}$ is a convex and bounded set,

(ii) $A$ is an $m \times n$ constraint matrix, $x$ is an $n$-dimensional vector of decision variable and $b \in R^m$,

(iii) $p \geq 2$,

(iv) $N_i\left(x\right) = \left(c^i\right)' x + d_i,\ D_i\left(x\right) = \left(e^i\right)' x + f_i, \forall i = \overline{1,p}$,

(v) $c^i, e^i \in R^n, d_i, f_i \in R, \forall i = \overline{1,p}$,

(vi) $\left(e^i\right)' x + f_i > 0, \forall i = \overline{1,p}, \forall x \in X$.

The term " max " being used in Problem (1) is for finding all weakly efficient and strongly efficient solutions in a maximization sense in terms of classic definitions [8].

Assumption (iv) is to fix notation for linear fractional objective function, i. e. linear nominators and denominators in criteria. Linearity is not a main condition under which our discussion takes place. Proposition 2 and the numerical example involve both this linearity in order to offer more specified description to the theoretical results and more intuitional facts to the practical results respectively.

To solve a multiple objective programming problem means to find a compromise solution. In this case, the fuzzy set theory proposes appropriate modeling tools for handling compromises. The goal of obtaining better solutions is connected to the goal of choosing appropriate fuzzy aggregation model depending on the application's specific nature.

## 3. On linguistic variables based solving methods

Imprecise aspirations of the decision-maker can be represented by structured linguistic variable. The concept of $(Z, \varepsilon)$-proximity will be used in the larger framework of the linguistic variables domain which leads with following membership functions:

$$C_j^{N_i}(x) = \begin{cases} 0 \quad , & \text{if} \quad N_i(x) < p_i^j \\ \dfrac{N_i(x) - p_i^j}{N_i^0 - p_i^j}, & \text{if} \quad p_i^j \le N_i(x) \le N_i^0 \end{cases}, \forall i = \overline{1,p}, \\ 0 \quad , & \text{if} \quad N_i(x) > N_i^0 \tag{2}$$

$$C_j^{D_i}(x) = \begin{cases} 0 \quad , & \text{if} \quad D_i(x) > s_i^j \\ \dfrac{s_i^j - D_i(x)}{s_i^j - D_i^0}, & \text{if} \quad D_i^0 \le D_i(x) \le s_i^j \\ 0 \quad , & \text{if} \quad D_i(x) < D_i^0 \end{cases}, \forall i = \overline{1,p} \tag{3}$$

or

$$C_j^{z_i}(x) = \begin{cases} 0 \quad , & \text{if} \quad z_i(x) < r_i^j \\ \dfrac{z_i(x) - r_i^j}{z_i^0 - r_i^j}, & \text{if} \quad r_i^j \le z_i(x) \le z_i^0 \\ 0 \quad , & \text{if} \quad z_i(x) > z_i^0 \end{cases}, \forall i = \overline{1,p}. \tag{4}$$

where $N_i^0$, $D_i^0$ and $z_i^0$ $\left(\forall i = \overline{1,p}\right)$ represent the maximal value of nominator $N_i(x)$, the minimal value of denominator $D_i(x)$ and the maximal value of linear fractional functions $z_i(x)$ on the set $X$, while $p_i^j$, $s_i^j$, $r_i^j$ $(j = 1, 2, 3)$ are the thresholds beginning with which values $N_i(x)$, $D_i(x)$ and $z_i(x)$ are (quite close, close, very close) acceptable.

When membership functions (2)-(3) are used an aggregation of them are made later in order to obtain a membership function for objective functions. These membership functions are better to be used than membership functions (4) because of linearity. Despite of this, an obtained membership function could loose essential properties of corresponding objective function if the aggregation operator is not well selected.

The aim of this paper is to analyze some aspects which must be taken into consideration when coefficients' selection in a linear combination of linear membership function is made. Next we will focus on one single objective function. Consequently, above $i$ indexes will not be necessary to be used and we will eliminate them in order to avoid complicated formulas. Thresholds' $j$ indexes will be also eliminated. Next we work with following membership functions.

$$C^N(x) = \begin{cases} 0 \quad , & \text{if} \quad N(x) < p \\ \dfrac{N(x) - p}{N^0 - p}, & \text{if} \quad p \le N(x) \le N^0 \\ 0 \quad , & \text{if} \quad N(x) > N^0 \end{cases} ,$$

for fuzzyfing nominator's maximization goal and

$$C^D(x) = \begin{cases} 0 \quad , & \text{if} \quad D(x) > s \\ \dfrac{s - D(x)}{s - D^0}, & \text{if} \quad D^0 \le D(x) \le s \\ 0 \quad , & \text{if} \quad D(x) < D^0 \end{cases}$$

for fuzzyfing denominator's minimization goal.

When a linear aggregation $\mu(x) = wC^N(x) + w'C^D(x)$ is made new membership function $\mu(x)$ would be better to verify hypothesis (5) in order to retain all essential properties of initial linear fractional function $z(x)$ from the optimization point of view.

$$\forall x^1, x^2 \in X, z\left(x^1\right) > z\left(x^2\right) \text{ then } \mu\left(x^1\right) > \mu\left(x^2\right) \tag{5}$$

In literature hypothesis (5) is stated to be verified or it is replaced by equivalent hypotheses. In [9] it is proved that coefficients $w$ and $w'$ must verified $k\underline{A} < w'/w < k\overline{A}$ where

$$\overline{A} = \min\left\{\frac{N(x^1) - N(x^2)}{D(x^1) - D(x^2)} \mid D\left(x^1\right) < D\left(x^2\right), \frac{N(x^1)}{D(x^1)} < \frac{N(x^2)}{D(x^2)}, x^1, x^2 \in X\right\}$$

and

$$\underline{A} = \max\left\{\frac{N(x^1) - N(x^2)}{D(x^1) - D(x^2)} \mid D\left(x^1\right) > D\left(x^2\right), \frac{N(x^1)}{D(x^1)} < \frac{N(x^2)}{D(x^2)}, x^1, x^2 \in X\right\}.$$

Using transformation $y = x^1 - x^2, x^1, x^2 \in X$ following equivalent formulas (6) to calculate $\overline{A}$ and $\underline{A}$ are obtained.

$$\overline{A} = \min\left\{\tfrac{c'y}{e'y} \mid e'y < 0, \tfrac{c'y}{d'y} > z\left(x^2\right)\right\}, \quad \underline{A} = \max\left\{\tfrac{c'y}{e'y} \mid e'y > 0, \tfrac{c'y}{e'y} < z\left(x^2\right)\right\}. \tag{6}$$

Because of propositions below we can conclude that hypothesis (5) and its equivalent forms could give an empty range for $w'/w$.

Proposition 1 *Hypothesis (5) can be verified by $z(x)$ and $\mu(x)$ if and only if*

$$\frac{\partial z}{\partial x_i} \cdot \frac{\partial \mu}{\partial x_i} \geq 0 \text{ for each } i = 1, 2, ..., n.$$

| Marginal points | $f_1$ | $N_1$ | $p_1$ | $D_1$ | $s_1$ | $f_2$ | $N_2$ | $p_1$ | $D_2$ | $s_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $(0,0)$ | $-0.142$ | $-1$ | 4 | 7 | 3 | $-0.5$ | $-2$ | 8 | 4 | 7 |
| $(6,0)$ | 5 | 5 | 4 | 1 | 3 | 2.5 | 10 | 8 | 4 | 7 |
| $(6,1)$ | 1.6 | 8 | 4 | 5 | 3 | 2 | 12 | 8 | 6 | 7 |

Table 1: Thresholds of the objective functions

Proposition 2 *A non-empty range for $w'/w$ to verifying hypothesis (5) exists if and only if*

$$h_i(x) = (c_i e' - e_i c')\, x + c_i f - e_i d, x \in X, i = 1, 2, ..., n$$

*doesn't change its sign over $X$, for each index $i$.*

Proposition 1 is a consequence of monotony from the $1-$dimensional case. In fact, hypothesis (5) states monotony along any direction. Proposition 2 is a consequence of computing partial derivatives. Derivatives of linear function $\mu$ are constant and the sign of derivatives of linear fractional function $z$ is stated by $h_i$.

$$\frac{\partial}{\partial x_i}\left(\frac{c'x+d}{e'x+f}\right) = \frac{(c_i e' - e_i c')\, x + c_i f - e_i d}{(e'x+f)^2}$$

$$\frac{\partial}{\partial x_i}\left[\alpha\,(c'x+d) + \beta\,(e'x+f)\right] = \alpha c_i + \beta e_i$$

Coefficients $\alpha$ and $\beta$ involve coefficients $w$ and $w'$, thresholds $p$ and $s$ and marginal solutions $N^0$ and $D^0$ of nominator and denominator respectively.

Returning to MOLFPP we have to conclude that any solving method can be improved by choosing proper membership functions for each criterion. An $n-$dimensional MOLFPP can be considered as being $2n-$dimensional MOLPP (multiple objective linear programming problem) if all fractional objective functions in MOLFPP come from 2−dimensional models. Otherwise, for fractional objective functions it is better to consider linguistic variable based membership functions which can retain their more essential properties.

## 4. Computational results

In this section, the example considered in [3] and [9] is discussed.

Example 3

$$\max\left(z_1(x) = \frac{x_1 + x_2 - 1}{-x_1 + 2x_2 + 7}, z_2(x) = \frac{2x_1 + x_2 - 2}{x_2 + 4}\right) \tag{7}$$

subject to

$$\begin{aligned} &-x_1 + 3x_2 \leq 0, \\ &x_1 \leq 6, \\ &x_1, x_2 \geq 0. \end{aligned} \tag{8}$$

The single efficient point of Problem (7)-(8) is $x^{opt} = (6, 0)$. Both objective functions reach in this point their optimum, independently one from another, on the same feasible region.

| $w_1$ | $w_1'$ | $w_1'/w_1$ | $w_2$ | $w_2'$ | $w_2'/w_2$ | Remarks |
|---|---|---|---|---|---|---|
| 0.035 | 0.465 | 13.285 | 0.475 | 0.025 | 5.263 | Efficient solution |
| 0.991 | 0.007 | 7.063 | 0.001 | 0.001 | 1 | Non-efficient solution |

Table 2: Efficiency established by different sets of aggregation coefficient values

Marginal solution of functions $z_1, z_2$ and of their nominators and denominators, marginal points of the feasible set $X$ and thresholds $p_1, p_2, s_1, s_2$ are described in Table 1. Optimizing (6) for different values of the aggregation coefficients the obtained results are written in Table 2 (in terms of efficiency).

| $i$ | $\overline{A}_i$ | $\underline{A}_i$ | Range for $w_i'/w_i$ | Remarks |
|---|---|---|---|---|
| 1 | 1.46189 | 4.98327 | $[3.334, 1]$ | $\emptyset$ |
| 2 | 1.47488 | 2.49800 | $[1.875, 1]$ | $\emptyset$ |

Table 3: Emptiness of aggregation coefficients intervals

Even the range of $w'/w$ is empty selecting different values for $w', w$ efficient or non-efficient solution can be obtained as it can be seen in Table 3.

| $i$ | $h_i^1(x)$ | Remarks |
|---|---|---|
| 1 | $3x_2 + 6$ | $\geq 0, \forall x \in X$ |
| 2 | $-3x_1 + 9$ | $\leq 0$ for $x_1 \geq 3$, $\geq 0$ for $x_1 \leq 3$ |

Table 4: Derivatives of the first objective function

Information from Table 4 states that hypothesis (5) couldn't be verified for any set of aggregation coefficients $w, w'$.

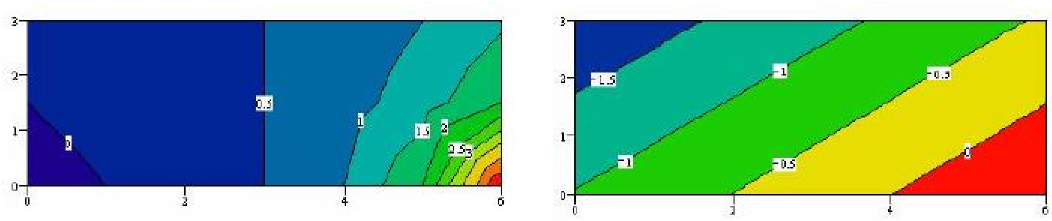

Figure 1: Fractional objective function and its linear membership function

In Figure 1 comparative graphical representations of the first fractional objective function and its linear membership function can be viewed. It is obvious that their behavior cannot be similar in the vicinity of the marginal point $(6, 0)$.

In literature all solving methods for Problem (7)-(8) claim hypothesis (5) to be verified. Instead of that the following counter-example (see Table 5) proves that the mentioned hypothesis is not generally satisfied. It also means that efficient solutions could be obtained even when the hypothesis is not verified.

## 5. Conclusions

Advantages of fuzzy models in practical use and general methods to solve fuzzy programming models were synthesized so far ([1, 4, 7]). Fuzzy approaches to solve deterministic problems were developed in recent literature.

We have addressed linguistic variable based method for solving linear fractional programming problems. Two propositions were formulated related to choosing possibilities of aggregation coefficients for fractional criteria' membership functions.

Computational results were developed in order to highlight theoretical remarks related to membership functions' for efficiency needed properties: hypothesis (5) is essential for a perfect behavior of the membership function; hypothesis (5) is not necessary in obtaining efficient points in multiple criteria optimization; when the hypothesis (5) is claimed to be satisfied a membership function for global fractional objective function has to be chosen instead of an aggregation of nominator and denominator membership functions.

| $x^i$ | $(2, 0.5)$ | $(2, 0.4)$ | Remarks |
|---|---|---|---|
| $Z(x^i)$ | 0.25 | 0.241 | $Z(x^1) > Z(x^2)$ |
| $\mu(x^i)$ | $-0.719$ | $-0.674$ | $\mu(x^1) < \mu(x^2)$ |

Table 5: Efficiency instead of unsatisfied claimed hypothesis

We have concluded that any solving method can be improved by choosing proper membership functions for each criterion in order to retain more essential properties of models. Also, an $n-$dimensional fractional problem it is better to be considered as being $2n-$dimensional linear problem if all fractional objective functions come from $2-$dimensional models.

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