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DIFFERENTIAL SUBORDINATIONS DEFINED BY USING SĂLĂGEAN DIFFERENTIAL OPERATOR AT THE CLASS OF MEROMORPHIC FUNCTIONS

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Abstract.By using the Sălăgean differential operator $D^n f(z)$, $z \in U$ (Definition 1), at the class of meromorphic functions we obtain some new differential subordination.

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1.Introduction and preliminaries

Denote by U the unit disc of the complex plane:

$$U = \{z \in \mathbf{C} : \ |z| < 1\},$$

and

$$\dot{U} = U - \{0\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U.

We let

$$A_n = \{f \in \mathcal{H}(U), \ f(z) = a + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, \ z \in U\}$$

with $A_1 = A$.

Let $\Sigma_{m,k}$ denote the class of functions in $\dot{U}$ of the form

$$f(z) = \frac{1}{z^m} + a_k z^k + a_{k+1}z^{k+1} + \dots, \ m \in \mathbf{N}^* = \{1, 2, 3, \dots\}$$

k integer, $k \geq -m + 1$, which are regular in the punctual disc $\dot{U}$. If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

A function $f \in \mathcal{H}(U)$ is said to be convex if it is univalent and $f(U)$ is a convex domain. It is well known that the function f is convex if and only if

$$f'(0) \neq 0 \text{ and } \mathrm{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > 0, \text{ for } z \in U.$$

We let

$$K = \left\{ f \in A,\ \mathrm{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > 0,\ z \in U \right\}.$$

In order to prove the new results, we use the following results.

Lemma A. (Hallenbeck and Ruscheweyh [1, p.71]) *Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbf{C}^*$ be a complex with* $\mathrm{Re}\gamma \geq 0$. *If $p \in \mathcal{H}(U)$, with $p(0) = a$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z)$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt.$$

The function q is convex and is the best (a, n)-dominant.

Lemma B. [1, p.66, Corollary 2.6.g.2] *Let $f \in A$ and F is given by*

$$F(z) = \frac{2}{z} \int_0^z f(t) dt.$$

If

$$\mathrm{Re}\frac{zf''(z)}{f'(z)} + 1 > -\frac{1}{2}, z \in U$$

then

$$F \in K.$$

For the case when $F(z)$ has a more elaborate form, Lemma B can be rewritten in the following form:

Lemma C. *Let $f \in A$, $\gamma > 1$ and F is given by*

$$F(z) = \frac{1+\gamma}{z^{\frac{1}{\gamma}}} \int_0^z f(t) t^{\frac{1}{\gamma}-1} dt.$$

If

$$\mathrm{Re} \frac{zf''(z)}{f'(z)} + 1 > -\frac{1}{2}, z \in U$$

then

$$F \in K.$$

Definition 1. [2]*For $f \in A$ and $n \in \mathbf{N}^* \cup \{0\}$ the operator $D^n f$ is defined by*

$$D^0 f(z) = f(z)$$
$$D^{n+1} f(z) = z[D^n f(z)]', \ z \in U.$$

Remark 1. If $f \in A$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \ z \in U$$

then

$$D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j, \ z \in U.$$

2.Main results

Theorem 1.*Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, which verifies the inequality:*

$$\mathrm{Re} \left[\frac{zh''(z)}{h'(z)} + 1 \right] > -\frac{1}{2(m+k)}, \ z \in U. \tag{1}$$

If $f \in \Sigma_{m,k}$ and verifies the differential subordination

$$[D^{n+1}(z^{m+1} f(z))]' \prec h(z), \ z \in U \tag{2}$$

then

$$[D^n z^{m+1} f(z)]' \prec g(z), \ z \in U$$

where

$$g(z) = \frac{1}{(m+k)z^{\frac{1}{m+k}}} \int_0^z h(t) t^{\frac{1}{m+k}-1} dt. \tag{3}$$

The function g is convex and is the best $(1, m+k)$ dominant.

Proof. By using properties of the operator $D^n f$ we have

$$D^{n+1}(z^{m+1} f(z)) = z[D^n(z^{m+1} f(z))]', \ z \in U. \tag{4}$$

Differentiating (4), we obtain

$$[D^{n+1}(z^{m+1} f(z))]' = [D^n(z^{m+1} f(z))]' + z[D^n(z^{m+1} f(z))]'', \ z \in U. \tag{5}$$

If we let

$$p(z) = [D^n(z^{m+1} f(z))]', \ z \in U, \tag{6}$$

then (5) becomes

$$[D^{n+1}(z^{m+1} f(z))]' = p(z) + zp'(z), \ z \in U. \tag{7}$$

Using (7), subordination (2) is equivalent to

$$p(z) + zp'(z) \prec h(z), \ z \in U, \tag{8}$$

where

$$p(z) = [D^n(z^{m+1} f(z))]' = \left[z + \sum_{j=m+k+1}^{\infty} a_j j^n z^j \right]'$$
$$= 1 + a_{m+k+1}(m+k+1)^n z^{m+k} + \dots$$

By using Lemma A, for $\gamma = 1$, $n = m+k$, we have

$$p(z) \prec g(z) \prec h(z),$$

where

$$g(z) = \frac{1}{(m+k)z^{\frac{1}{m+k}}} \int_0^z h(t) t^{\frac{1}{m+k}-1} dt, \ z \in U$$

and is the best $(1, m+k)$ dominant.

By applying Lemma C for the function given by (3) and function h with the property in (1) for $\gamma = m+k > 1$ we obtain that function g is convex.

THEOREM 2.*Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, which verifies the inequality*

$$\mathrm{Re}\left[\frac{zh''(z)}{h'(z)} + 1\right] > -\frac{1}{2(m+k)}, z \in U.$$

If $f \in \Sigma_{m,k}$ and verifies the differential subordination

$$[D^n(z^{m+1}f(z))]' \prec h(z), \ z \in U \tag{9}$$

then

$$\frac{D^n(z^{m+1}f(z)))}{z} \prec g(z), \ z \in U$$

where

$$g(z) = \frac{1}{(m+k)z^{\frac{1}{m+k}}} \int_0^z h(t)t^{\frac{1}{m+k}-1}dt, \ z \in U.$$

The function g is convex and is the best $(1, m+k)$ dominant.

Proof. We let

$$p(z) = \frac{D^n(z^{m+1}f(z))}{z}, \ z \in U \tag{10}$$

and we obtain

$$D^n(z^{m+1}f(z)) = zp(z), \ z \in U. \tag{11}$$

By differentiating (11), we obtain

$$[D^n(z^{m+1}f(z))]' = p(z) + zp'(z), \ z \in U.$$

Then (9) becomes

$$p(z) + zp'(z) \prec h(z),$$

where

$$p(z) = \frac{z + \sum_{j=m+k+1}^{\infty} a_j j^n z^j}{z} = 1 + p_{m+k+1}z^{m+k} + \dots, \ z \in U.$$

By using Lemma A, for $\gamma = 1$, $n = m+k$, we have

$$p(z) \prec g(z) \prec h(z),$$

where

$$g(z)=\frac{1}{(m+k)z^{\frac{1}{m+k}}}\int_0^z h(t)t^{\frac{1}{m+k}-1}dt,\ z\in U,$$

and g is the best $(1,m+k)$-dominant.

By applying Lemma C for function g given by (3) and function h with the property in (1) for $\gamma=m+k>1$, we obtain that function g is convex.

References

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