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AN INTEGRAL UNIVALENT OPERATOR II

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Abstract.In this paper we present the univalence conditions for the operator

$$G_{\alpha,n}(z)=\left((n(\alpha-1)+1)\int_0^z g_1^{\alpha-1}(t)\dots g_n^{\alpha-1}(t)\,\mathrm{d}t\right)^{\frac{1}{n(\alpha-1)+1}}$$

in the unit disc.

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Introduction

We consider U be the unit disc and denote by $H(U)$ the class of holomorphic functions in U. Let the set of analytic functions

$$A_n=\left\{f\in H(U):f(z)=z+a_{n+1}z^{n+1}+\dots\right\} \tag{1}$$

For $n=1$ obtain $A_1=\{f\in H(U):f(z)=z+a_2z^2+\dots\}$, and denote $A_1=A$. Let S the class of regular and univalent functions $f(z)=z+a_2z^2+\dots$ in U, which satisfy the condition $f(0)=f'(0)-1=0$.

Ozaki and Nunokawa proved in [3] the following results:

Theorem 1.*If we assume that $g\in A$ satisfies the condition*

$$\left|\frac{z^2f'(z)}{f^2(z)}-1\right|<1, z\in U \tag{2}$$

then f is univalent in U.

The Schwartz Lemma. *Let the analytic function g be regular in the unit disc U and $g(0)=0$. If $|g(z)|\le 1,\forall z\in U$, then*

$$|g(z)|\le|z|,\forall z\in U \tag{3}$$

and equality holds only if $g(z)=\varepsilon z$, where $|\varepsilon|=1$.

THEOREM 2.*Let α be a complex number with $\operatorname{Re}\alpha > 0$, and let $f = z + a_2z^2 + \dots$ be a regular function on U.*

If

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \forall z \in U \tag{4}$$

then for any complex number β with Re $\beta \geq$ Re α the function

$$F_\beta(z) = \left(\beta\int_0^z t^{\beta-1}f'(t)\,\mathrm{d}t\right)^{\frac{1}{\beta}} = z + \dots \tag{5}$$

is regular and univalent in U.

THEOREM 3.*Assume that $g \in A$ satisfies condition (2), and let α be a complex number with*

$$|\alpha - 1| \le \frac{\operatorname{Re}\alpha}{3}. \tag{6}$$

If

$$|g(z)| \le 1, \forall z \in U \tag{7}$$

then the function

$$G_\alpha(z) = \left(\alpha\int_0^z g^{(\alpha-1)}(t)\,\mathrm{d}t\right)^{\frac{1}{\alpha}} \tag{8}$$

is of class S.

MAIN RESULTS

THEOREM 4.*Let $g_i \in A$, for all $i = \overline{1,n}$, $n \in N^*$, satisfy the properties*

$$\left|\frac{z^2g_i'(z)}{g_i^2(z)} - 1\right| < 1, \forall z \in U, \forall i = \overline{1,n} \tag{9}$$

and $\alpha \in C$, with

$$|\alpha - 1| \le \frac{\operatorname{Re}\alpha}{3n}. \tag{10}$$

If $|g_i(z)| \le 1,\ \forall z \in U,\ \forall i = \overline{1,n}$, then the function

$$G_{\alpha,n}(z) = \left((n(\alpha-1)+1)\int_0^z g_1^{\alpha-1}(t)\dots g_n^{\alpha-1}(t)\,\mathrm{d}t\right)^{\frac{1}{n(\alpha-1)+1}} \tag{11}$$

is univalent.

Proof. From (11) $G_{\alpha,n}$ can be written:

$$G_{\alpha,n}(z) = \left((n(\alpha-1)+1)\int_0^z t^{n(\alpha-1)}\left(\frac{g_1(t)}{t}\right)^{\alpha-1}\cdots\left(\frac{g_n(t)}{t}\right)^{\alpha-1}\mathrm{d}t\right)^{\frac{1}{n(\alpha-1)+1}} \tag{12}$$

We consider the function

$$f(z) = \int_0^z \left(\frac{g_1(t)}{t}\right)^{\alpha-1}\cdots\left(\frac{g_n(t)}{t}\right)^{\alpha-1}\mathrm{d}t. \tag{13}$$

The function f is regular in U, and from (13) we obtain

$$f'(z) = \left(\frac{g_1(z)}{z}\right)^{\alpha-1}\cdots\left(\frac{g_n(z)}{z}\right)^{\alpha-1} \tag{14}$$

and

$$f''(z) = (\alpha-1)\sum_{k=1}^{n} A_k B_k \tag{15}$$

where A_k and B_k has the next form:

$$A_k = \left(\frac{g_k(z)}{z}\right)^{\alpha-2}\frac{zg_k'(z)-g_k(z)}{z^2} \tag{16}$$

and

$$B_k = f'(z)\cdot\left(\frac{z}{g_k(z)}\right)^{\alpha-1}. \tag{17}$$

Next we calculate the expresion $\frac{zf''}{f'}$.

$$\frac{zf''(z)}{f'(z)} = (\alpha-1)\cdot\sum_{k=1}^{n}\frac{zg_k'(z)-1}{g_k(z)} \tag{18}$$

The modulus

$$\left|\frac{zf''}{f'}\right| \tag{19}$$

can then be evaluated as

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|(\alpha-1)\cdot\sum_{k=1}^{n}\frac{zg_k'(z)-1}{g_k(z)}\right| \le \sum_{k=1}^{n}\left|(\alpha-1)\cdot\frac{zg_k'(z)-1}{g_k(z)}\right|. \tag{20}$$

Multiplying the first and the last terms of (20) with $\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}>0$, we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\alpha-1|\sum_{k=1}^{n}\left(\left|\frac{zg_k'(z)}{g_k(z)}\right|+1\right) \tag{21}$$

Applying the Schwartz Lemma and using (21), we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le |\alpha-1|\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\sum_{k=1}^{n}\left(\left|\frac{z^2g_k'(z)}{g_k^2(z)}-1\right|+2\right) \tag{22}$$

Since g_i satisfies the condition (2) $\forall i=\overline{1,n}$, then from (22) we obtain:

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 3n|\alpha-1|\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \le \frac{3n|\alpha-1|}{\operatorname{Re}\alpha}. \tag{23}$$

But $|\alpha-1|\le\frac{\operatorname{Re}\alpha}{3n}$ so from (10) we obtain that

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \tag{24}$$

for all $z\in U$ and the Theorem 2 implies that the function $G_{\alpha,n}$ is in the class S.

Corollary 5.*Let $g\in A$ satisfy (2) and α a complex number such that*

$$|\alpha-1|\le\frac{\operatorname{Re}\alpha}{3k}, k\in N^* \tag{25}$$

If $|g(z)|\le 1, \forall z\in U$, then the function

$$G_\alpha^k(z)=\left((k(\alpha-1)+1)\int_0^z g^{k(\alpha-1)}(t)\,\mathrm{d}t\right)^{\frac{1}{k(\alpha-1)+1}} \tag{26}$$

is univalent.

Proof.

We consider the functions

$$G_{\alpha}^{k}(z)=\left((k(\alpha-1)+1)\int_{0}^{z}t^{k(\alpha-1)}\left(\frac{g(t)}{t}\right)^{k(\alpha-1)}\mathrm{d}t\right)^{\frac{1}{k(\alpha-1)+1}} \tag{27}$$

and

$$f(z)=\int_{0}^{z}\left(\frac{g(t)}{t}\right)^{k(\alpha-1)}\mathrm{d}t. \tag{28}$$

The function f is regular in U. From (28) we obtain

$$f'(z)=\left(\frac{g(z)}{z}\right)^{k(\alpha-1)}$$

and

$$f''(z)=k(\alpha-1)\left(\frac{g(z)}{z}\right)^{k(\alpha-1)-1}\frac{zg'(z)-g(z)}{z^{2}}.$$

Next we have

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right|\leq\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}k|\alpha-1|\left(\left|\frac{zg'(z)}{g(z)}\right|+1\right),\forall z\in U. \tag{29}$$

Applying the Schwartz Lemma and using (29) we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right|\leq|\alpha-1|\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left(\left|\frac{z^{2}g'(z)}{g^{2}(z)}-1\right|+2\right). \tag{30}$$

Since g satisfies conditions (2) then from (30) and (25) we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right|\leq\frac{3k|\alpha-1|}{\operatorname{Re}\alpha}\left(1-|z|^{2\operatorname{Re}\alpha}\right)\leq\frac{3k|\alpha-1|}{\operatorname{Re}\alpha}\leq 1. \tag{31}$$

Now Theorem 2 and (31) imply that $G_{\alpha}^{k}\in S$.

Corollary 6. *Let $f,g\in A$, satisfy* (1) *and α the complex number with the property*

$$|\alpha-1|\leq\frac{\operatorname{Re}\alpha}{6}. \tag{32}$$

If $|f(z)| \leq 1, \forall z \in U$ *and* $|g(z)| \leq 1, \forall z \in U$, *then the function*

$$G_\alpha(z) = \left((2\alpha - 1)\int_0^z f^{(\alpha-1)}(t)\, g^{(\alpha-1)}(t)\, \mathrm{d}t\right)^{\frac{1}{2\alpha-1}} \tag{33}$$

is univalent.

Proof. In Theorem 4 we set $n = 2$, $g_1 = f$, $g_2 = g$.

REMARK. Theorem 4 is a generalization of Theorem 3.

REMARK. From Corollary 5, for $k = 1$, we obtain Theorem 3.

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