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IDEAL SLANT SUBMANIFOLDS IN COMPLEX SPACE FORMS

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Abstract. Roughly speaking, an ideal immersion of a Riemannian manifold into a space form is an isometric immersion which produces the least possible amount of tension from the ambient space at each point of the submanifold. Recently, B.-Y. Chen studied Lagrangian submanifolds in complex space forms which are ideal. He proved that such submanifolds are minimal. He also classified ideal Lagrangian submanifolds in complex space forms.
In the present paper, we investigate ideal Kaehlerian slant submanifolds in a complex space form. We prove that such submanifolds are minimal. We also obtain obstructions to ideal slant immersions in complex hyperbolic space and complex projective space.

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1. Chen invariants and Chen inequalities

One of the most fundamental problems in submanifold theory is the immersibility of a Riemannian manifold in a Euclidean space (or, more generally, in a space form). According to the well-known embedding theorem of Nash, every Riemannian manifold can be isometrically embedded in some Euclidean space with sufficiently high codimension. Thus Riemannian manifolds could always be regarded as Riemannian submanifolds. This would then yield the opportunity to use extrinsic help.

It is well-known that Riemannian invariants play the most fundamental role in Riemannian geometry. Riemannian invariants provide the intrinsic characteristics of Riemannian manifolds which affect the behavior in general of the Riemannian manifold.

One needs to introduce new types of Riemannian invariants, different from the known ones (e.g., Ricci curvature and scalar curvature) as well as to establish general optimal relationships between the main extrinsic invariants and the new type of intrinsic invariants of submanifolds.

Let M be an n-dimensional Riemannian manifold. We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $\{e_1, ..., e_n\}$ of the tangent space T_pM, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j). \tag{1}$$

Let L be a subspace of T_pM of dimension $r \geq 2$ and $\{e_1, ..., e_r\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of the r-plane section L is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r. \tag{2}$$

For an integer $k \geq 0$, we denote by $\mathcal{S}(n,k)$ the finite set consisting of unordered k-tuples $(n_1, \ldots, n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + ... + n_k \leq n$. Then $\mathcal{S}(n)$ is the union $\cup_{k\geq 0} \mathcal{S}(n,k)$.

B.-Y. Chen introduced in [2], [3] a new type of curvature invariants $\delta(n_1, ..., n_k)$ known as Chen invariants, defined as follows.

$$\delta(n_1, ..., n_k) = \tau - \inf\{\tau(L_1) + ... + \tau(L_k)\}, \tag{3}$$

where, at each point $p \in M$, $L_1, ..., L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, ..., k$.

He proved in [3] an optimal relationship between the Chen invariants $\delta(n_1, \ldots, n_k)$ and the squared mean curvature $\|H\|^2$, which we call Chen inequality, for an arbitrary submanifold in a real space form.

B.-Y. Chen also pointed out in [3] that the same result holds for totally real submanifolds in complex space forms.

Let $\widetilde{M}(4c)$ be an m-dimensional complex space form with constant holomorphic sectional curvature $4c$. We denote by J the complex structure of $\widetilde{M}(4c)$. The curvature tensor $\widetilde{R}$ of $\widetilde{M}(4c)$ is given by:

$$\widetilde{R}(X,Y)Z = c(g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ), \tag{4}$$

for any tangent vector fields X, Y, Z to $\widetilde{M}(4c)$.

An n-dimensional submanifold M of $\widetilde{M}(4c)$ is said to be a *slant submanifold* [1] if, for any $p \in M$ and any nonzero vector $X \in T_pM$, the angle between JX and the tangent space T_pM is constant which is denoted by θ. Moreover, if M is neither complex ($\theta = 0$) nor totally real ($\theta = \frac{\pi}{2}$), then M is called a proper slant submanifold.

We denote by h and A the second fundamental form and the shape operator, respectively, of M in $\widetilde{M}(4c)$.

The equations of Gauss and Codazzi are given respectively by

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,Z),h(Y,W)) + g(h(X,W),h(Y,Z)), \tag{5}$$

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z), \tag{6}$$

where X, Y, Z, W are tangent vector fields to M, and ∇h is defined by

$$(\nabla_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

For any vector field X tangent to M, we put $JX = PX + FX$, where PX and FX are the tangential and normal components of JX. Thus, P is an endomorphism of the tangent bundle TM.

We obtained inequalities satisfied by some Chen invariants $\delta'(n_1, ..., n_k)$ for slant submanifolds in a complex space form in [9].

We consider the Riemannian invariant

$$\delta'(n_1, ..., n_k) = \tau - \inf\{\tau(L_1) + ... + \tau(L_k)\},$$

where, at each point $p \in M$, $L_1, ..., L_k$ run over all k mutually orthogonal subspaces of T_pM invariant by the endomorphism P, such that $\dim L_j = n_j$, $j = 1, ..., k$. We set

$$b(n_1, ..., n_k) = \frac{n^2(n + k - 1 - \sum_{j=1}^{k} n_j)}{2(n + k - \sum_{j=1}^{k} n_j)},$$

$$d(n_1, ..., n_k) = \frac{1}{2}\left[n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\right].$$

Theorem 1.1. *Let M be an n-dimensional θ-slant submanifold of an m-dimensional complex space form $\widetilde{M}(4c)$. Then, for any $(n_1, ..., n_k) \in \mathcal{S}(n)$, we have*

$$\delta'(n_1, ..., n_k) \leq b(n_1, ..., n_k)\|H\|^2 + d(n_1, ..., n_k)c + \frac{3}{2}(n - \sum_{j=1}^{k} n_j)c\cos^2\theta. \quad (7)$$

Moreover, the equality holds at a point $p \in M$ if and only if there exist a tangent basis $\{e_1, ..., e_n\} \subset T_pM$ and a normal basis $\{e_{n+1}, ..., e_{2m}\} \subset T_p^{\perp}M$ such that, for any vector ξ normal to M at p, the shape operator A_ξ takes the following form

$$A_\xi = \begin{pmatrix} A_1^\xi & ... & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & ... & A_k^\xi & \\ & 0 & & \mu_\xi I \end{pmatrix}, \quad (8)$$

where I is an identity matrix and A_j^ξ is a symmetric $n_j \times n_j$ submatrix satisfying

$$\mathrm{tr}\,(A_1^\xi) = ... = \mathrm{tr}\,(A_k^\xi) = \mu_\xi. \quad (9)$$

A slant submanifold M of a complex space form $\widetilde{M}(4c)$ is called *ideal* if it satisfies the equality case of the inequality (7) identically for some $(n_1, ..., n_k) \in \mathcal{S}(n)$.

2. Minimality of ideal submanifolds and applications

In the following section, we will investigate n-dimensional Kaehlerian slant ideal submanifolds in an n-dimensional complex space form $\widetilde{M}(4c)$.

We recall a proper slant submanifold is *Kaehlerian slant* if the endomorphism P is parallel with respect to the Riemannian connection ∇. It is known (see [1]) that this condition is equivalent to

$$A_{FX}Y = A_{FY}X, \quad \forall X, Y \in \Gamma(TM). \quad (10)$$

Theorem 2.1. *Let M be an n-dimensional Kaehlerian slant submanifold of an n-dimensional complex space form $\widetilde{M}(4c)$. If M is an ideal submanifold, then it is minimal.*

As applications, we derive certain obstructions to ideal slant immersions. First, we state the following.

Proposition 2.2. *Every minimal slant submanifold of a hyperbolic complex space form is irreducible.*

Recall that the first normal space Im h_p and the relative null space Ker h_p of a submanifold M at a point $p \in M$ are the vector spaces defined respectively by

$$\text{Im } h_p = \text{sp } \{h(X,Y) |\, X, Y \in T_pM\},$$
$$\text{Ker } h_p = \{Z \in T_pM |\, h(X,Z) = 0, \forall X \in T_pM\}.$$

It is easily seen that the first normal space Im h_p and the relative null space Ker h_p of a Kaehlerian slant submanifold M in a complex space form $\widetilde{M}(4c)$ are related by $(\text{Im } h_p)^{\perp} = F(\text{Ker } h_p)$.

Proposition 2.2 implies the following

Theorem 2.3. *There do not exist n-dimensional ideal Kaehlerian slant submanifolds in an n-dimensional complex hyperbolic space whose first normal bundle is full.*

On the other hand, there do exist n-dimensional ideal Kaehlerian slant submanifolds in the complex Euclidean space $\mathbf{C}^n$ with full first normal bundle. In fact, we have the following.

Theorem 2.4. *Let M be an n-dimensional Kaehlerian slant submanifold in $\mathbf{C}^n$ with full first normal bundle. Then M is ideal if and only if, locally, M is the Riemannian product of some minimal Kaehlerian slant submanifolds M_j, $j = 1, ..., k$, with full first normal bundle.*

We state a theorem of characterization of ideal Kaehlerian slant submanifolds in the complex Euclidean space.

Theorem 2.5. *Let M be an n-dimensional Kaehlerian slant submanifold of the complex Euclidean space $\mathbf{C}^n$ such that Im $h_p \neq T_p^{\perp}M$, at each point $p \in M$. Then M is ideal if and only if M is a ruled minimal submanifold.*

On the other hand, combining a very recent result of I. Salavessa [10] and Theorem 2.1 of [7], we have the following non-existence result.

THEOREM 2.6. *There do not exist n-dimensional ideal Kaehlerian slant submanifolds in the complex projective space $P^n(\mathbf{C})$.*

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