AN ALTERNATIVE METHOD FOR SOLVING DIRECT AND INVERSE STOKES PROBLEMS

A. E. Curteanu, L. Elliott, D. B. Ingham & D. Lesnic

Abstract. The solution of the equations governing the two-dimensional slow viscous fluid flow is analysed using a novel technique based on a Laplacian decomposition instead of the more traditional approaches based on the biharmonic streamfunction formulation or the velocity-pressure formulation. This results in the need to solve three Laplace equations for the pressure and two auxiliary harmonic functions, which in turn stems from Almansi decomposition. These equations, which become coupled through the boundary conditions, are numerically solved using the boundary element method (BEM). Numerical results for both direct and inverse problem are presented and discussed considering a benchmark test example in a non-smooth geometry.

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1. Introduction

The Stokes equations describe fluid flows at low Reynolds number when the viscous forces dominate over the inertial forces. These laminar flows are important in flows in pipes and open channels, seepage of water and oil underground, for very small bodies, such as small spheres, in a highly viscous fluid, in the theory of lubrication, etc. During the last few decades there has been interest in applying the BEM for the solution of Stokes flow problems, e.g. see Higdon (1985), Zeb *et al.* (1998), Lesnic *et al.* (1999) and Zeb *et al.* (2000).

2. Mathematical formulation

The Navier-Stokes equations of motion are mathematical statements of the dynamical conditions within a viscous fluid flow, and as such are expected to accurately apply to each and every problem involving a laminar viscous fluid flow. However, due to the mathematical complexity of the equations, it is well known that the general solution of these equations is not possible. Therefore in order to construct tractable mathematical models of the fluid flow systems, it is necessary to resort to a number of simplifications. One of these simplifications occurs when viscous forces are of a higher-order of magnitude as compared to the inertial forces. Consequently, one may drop the inertia terms from the steady Navier-Stokes equations to obtain the Stokes equation:

$$\mu\overline{\nabla}^2\ \underline{\overline{q}} = \overline{\nabla}\ \overline{P} \tag{1}$$

where $\underline{\overline{q}}$ is the fluid velocity vector, μ the viscosity of the fluid and $\overline{P}$ the pressure. Non-dimensionalizing equation (1), using typical velocity and length scales U_0 and L, respectively, and defining $\underline{\overline{r}} = L\underline{r}$, $\underline{\overline{q}} = \overline{U_0}\underline{q}$ and $\overline{P} = \frac{\mu\overline{U_0}}{L}P$, gives

$$\nabla^2\ \underline{q} = \nabla\ P \tag{2}$$

Since the fluid flow is assumed to be incompressible, we also have the continuity equation

$$\nabla \cdot \underline{q} = 0 \tag{3}$$

Differentiating the x and y components of equation (2) with respect to x and y, respectively, then adding these together and using the continuity equation, results in $\nabla^2 P = 0$.
In order to simplify equations (2) and (3), the following formulation for the velocity components are introduced:

$$u = f + \frac{x}{2}P \tag{4}$$
$$v = g + \frac{y}{2}P \tag{5}$$

resulting in f and g being solutions of the Laplace equation, namely $\nabla^2 f = 0$ and $\nabla^2 g = 0$. It is the above substitutions that have reduced the Stokes equations (2) and (3) to the solution of three Laplace equations. A similar idea has been proposed by Jaswon (1994) for solving the biharmonic equation.

BEM - Integral equation

In this paper, the development of the BEM for discretizing the Laplace equation is the classical approach, see Brebbia *et al.* (1984), and it is based on using the fundamental solution for the Laplace equation and Green's identities. Thus, e.g. equation $\nabla^2 f = 0$ may be recast as follows:

$$\eta(q)f(q) = \int_{\Gamma} \{f'(q')G(q,q') - f(q')G'(q,q')\} d\Gamma_{q'} \tag{6}$$

where

(i) $q \in \Omega \cup \Gamma$ and $q' \in \Gamma = \partial\Omega$, and Γ is the boundary of the domain Ω,

(ii) $d\Gamma_{q'}$ denotes the differential increment of Γ at q',

(iii) $\eta(q) = 1$ if $q \in \Omega$, and $\eta(q) =$ the internal angle between the tangents to Γ on either side of q divided by 2π if $q \in \Gamma$,

(iv) G is the fundamental solution for the Laplace equation which in two-dimensions is given by

$$G(q,q') = -\frac{1}{2\pi} ln|q - q'| \tag{7}$$

(v) G', f' are the outward normal derivatives of G, f and g, respectively.

In practice, the integral equation (6) can rarely be solved analytically and thus some form of numerical approximation is necessary. Based on the BEM, we subdivide the boundary Γ into a series of N elements Γ_j, $j = \overline{1,N}$, and approximate the functions f and f' at the collocation points of each boundary element Γ_j by f_j and f'_j, respectively. Thus the boundary integral equation (6) can be approximated as follows:

$$\eta(q_i)f(q_i) = \sum_{j=1}^{N} [f'_j \int_{\Gamma_j} G(q_i,q')d\Gamma_{q'} - f_j \int_{\Gamma_j} G'(q_i,q')d\Gamma_{q'}] \tag{8}$$

where $q' \in \Gamma$ and q_i is the centroid node of each element Γ_i when the BEM with constant elements (CBEM) is used, the end points of each element when

using BEM with linear elements (LBEM) and the points are placed at a quarter of each segments length from the boundary edges when using BEM with discontinuous linear elements (DLBEM).
For the CBEM, we introduce

$$A_{ij} = \int_{\Gamma_j} G(q_i, q')d\Gamma_{q'} \tag{9}$$

$$B_{ij} = \int_{\Gamma_j} G'(q_i, q')d\Gamma_{q'} + \eta(q_i)\delta_{ij} \tag{10}$$

where δ_{ij} is the Kronecker delta and then equation (8) reduces to the following linear algebraic equations:

$$\sum_{j=1}^{N}[A_{ij}f'_j - B_{ij}f_j] = 0 \tag{11}$$

The values of the integral coefficients (9) and (10) depend solely on the geometry of the boundary Γ and can be calculated analytically, see for example Mera *et al.* (2001).

Now the mathematical formulation of the problem can be reduced to a system of $3N$ equations in $6N$ unknowns, i.e.

$$\begin{cases} A\boldsymbol{f'} - B\boldsymbol{f} = 0 \\ A\boldsymbol{g'} - B\boldsymbol{g} = 0 \\ A\boldsymbol{P'} - B\boldsymbol{P} = 0 \end{cases} \tag{12}$$

and $3N$ appropriate boundary conditions. Due to some practical reasons it may be impossible to measure certain quantities, such as one, or both, of the velocity components on particular sections of the boundary, and instead one has to work with extra information from elsewhere.

Direct problem

First, we consider the case in which the two velocity components, namely u and v, are specified over the whole boundary of the solution domain. Thus the two boundary conditions, together with the three Laplace equations (12), provide a system of $5N$ equations with $6N$ unknowns. In order to acquire the extra boundary condition to solve this higher-order system of equations, additional information is sought from within that already prescribed. Thus, the remaining N equations necessary to solve the system of equations are appropriately obtained by enforcing the continuity equation (3) on the boundary, i.e. using the normal derivative of the normal component of velocity, which in the case of a square domain can be expressed directly in terms of the tangential derivative of the known tangential component of the velocity. This results in an equation connecting f', g', P and P' on the boundary.

Dividing the boundary such as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where $\Gamma_1 = \{(x,y)| -\frac{1}{2} \leq x \leq \frac{1}{2}, y = -\frac{1}{2}\}$, $\Gamma_2 = \{(x,y)| x = \frac{1}{2}, -\frac{1}{2} \leq y \leq \frac{1}{2}\}$, $\Gamma_3 = \{(x,y)| -\frac{1}{2} \leq x \leq \frac{1}{2}, y = \frac{1}{2}\}$, $\Gamma_4 = \{(x,y)| x = -\frac{1}{2}, -\frac{1}{2} \leq y \leq \frac{1}{2}\}$, the problem can be described mathematically by (12) and the following boundary conditions:

$$\begin{cases} \boldsymbol{f} + \frac{x}{2}\boldsymbol{P} = \boldsymbol{u}^{(an)} & \text{on } \Gamma \\ \boldsymbol{g} + \frac{y}{2}\boldsymbol{P} = \boldsymbol{v}^{(an)} & \text{on } \Gamma \\ \boldsymbol{f'} + \nu_1\boldsymbol{P} + \frac{x}{2}\boldsymbol{P'} = \boldsymbol{u'}^{(an)} & \text{on } \Gamma_2 \cup \Gamma_4 \\ \boldsymbol{g'} + \nu_2\boldsymbol{P} + \frac{y}{2}\boldsymbol{P'} = \boldsymbol{v'}^{(an)} & \text{on } \Gamma_1 \cup \Gamma_3 \end{cases} \tag{13}$$

where $\frac{x}{2}$, $\frac{y}{2}$, $\nu_1 = \frac{\partial(x/2)}{\partial n}$, $\nu_2 = \frac{\partial(y/2)}{\partial n}$ are the matrices $\delta_{ij}\frac{x_{(j)}}{2}$, $\delta_{ij}\frac{y_{(j)}}{2}$, $\delta_{ij}\nu_{1(j)}$ and $\delta_{ij}\nu_{2(j)}$, respectively .

Since u and v are given on the boundary, u' and v' can be clearly obtained analytically by using equations $\frac{\partial u}{\partial n} = \pm\frac{\partial u}{\partial x} = \mp\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial n} = \pm\frac{\partial v}{\partial y} = \mp\frac{\partial u}{\partial x}$, respectively.

In a generic form, the system of equations (12) and (13) can be rewritten as

$$\mathbb{A}\boldsymbol{x} = \boldsymbol{b} \tag{14}$$

where $\mathbb{A}$ is a known $6N \times 6N$ matrix which includes the matrices A and B, $\boldsymbol{x}$ is a vector of $6N$ unknowns which includes the vectors $\boldsymbol{f}$, $\boldsymbol{f'}$, $\boldsymbol{g}$, $\boldsymbol{g'}$, $\boldsymbol{P}$ and $\boldsymbol{P'}$, and $\boldsymbol{b}$ is a vector of $6N$ knowns which includes $\boldsymbol{u}$, $\boldsymbol{v}$ and the derivatives of velocity. This system of equations can be solved using, for example, a Gaussian elimination procedure. Then, using the calculated boundary data, interior solutions for the two harmonic functions and the velocity can be determined explicitly using equation (6), see Brebbia *et al.* (1984). We note that when using the DLBEM the system of equations (14) has $12N$ equations and $12N$ unknowns.

Since the combination between the harmonic functions is not unique, see Jaswon (1994), we have to establish the appropriate constraint in order to overcome this problem.

Uniqueness of the solution

Let f_0, g_0 and P_0 be solutions to the problem. Given $u = f_0 + \frac{x}{2}P_0$ and $v = g_0 + \frac{y}{2}P_0$, then the transformation $P = P_0 + a$ with $\nabla^2 P = 0$, implies the corresponding transformations $f = f_0 - \frac{ax}{2}$ and $g = g_0 - \frac{ay}{2}$ which require $\nabla^2(\frac{ax}{2}) = 0$ and $\nabla^2(\frac{ay}{2}) = 0$. These two last equalities are clearly satisfied. Moreover, substituting accordingly f, f', g, g', P and P' from the above transformations into the last two equations of the system (13) then these equations are also satisfied. Thus, the unique solution of the system (14) is obtained provided that we impose the value of the unknown harmonic function P at one point on Γ.

Imposing the uniqueness condition, the system of linear equations (14) is now over-determined and the solution vector $\boldsymbol{x}$

$$\boldsymbol{x} = (\mathbb{A}^{tr}\mathbb{A})^{-1}\mathbb{A}^{tr}\boldsymbol{b} \tag{15}$$

is obtained on applying the least squares approach, see Strang (1976).

Inverse problem

Next, we investigate an inverse Stokes problem in which it is not possible to specify two conditions at all points of the boundary $\Gamma = \Gamma_0 \cup \Gamma^*$ of the solution domain. Consequently, a part of the boundary remained under-specified and in order to compensate for this under-specification we used the pressure as extra information on another part of the boundary, which gives rise to a portion of boundary being over-specified.

In the inverse formulation of the Stokes problem considered, both components of the fluid velocity, namely u and v, are specified on the section of the boundary Γ^*, whilst only u is given on Γ_0. However, this under-specification of the boundary conditions on Γ_0 is compensated by the additional pressure measurements over Γ^*.

Suppose that the number of constant boundary elements N on Γ is such that N_0 belongs to $\Gamma_0 = \Gamma_1$ and $N - N_0$ to $\Gamma^* = \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, in a generic form the inverse problem reduces to the following system of equations:

$$\mathbb{C}\boldsymbol{x} = \boldsymbol{d} \tag{16}$$

where, $\mathbb{C}$ is a known $(6N - N_0) \times (5N + N_0)$ matrix which includes the matrices A and B, $\boldsymbol{x}$ is a vector of $5N + N_0$ unknowns which includes the N vectors $\boldsymbol{f}$, $\boldsymbol{f'}$, $\boldsymbol{g}$, $\boldsymbol{g'}$, $\boldsymbol{P'}$ and the N_0 vector $\boldsymbol{P}$, and $\boldsymbol{d}$ is a vector of $6N - N_0$ knowns which includes $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{P}^{(n)}$ and the derivatives of velocity and the specified numerical boundary pressure. We note that when using DLBEM the system (16) has $12N - 2N_0$ equations and $10N + 2N_0$ unknowns. Due to the ill-posed nature of the inverse problem, the system of linear equations obtained is ill-conditioned and cannot be solved using a direct approach, e.g. Gaussian elimination method or a simple inversion scheme. Furthermore, as the least squares solution also fails, the Tikhonov regularization method is used in order to solve the inverse problem.

The zeroth-order regularization method modifies the least squares approach and finds a stable approximate numerical solution of the system of equations (16) which is given by, see Tikhonov and Arsein (1977),

$$\boldsymbol{x}_\lambda = (\mathbb{C}^t\mathbb{C} + \lambda\mathbb{I})^{-1}\mathbb{C}^t\boldsymbol{d} \tag{17}$$

where $\mathbb{I}$ is the identity matrix and $\lambda > 0$ is the regularization parameter, which controls the degree of smoothing applied to the solution and whose choice may be based on the L-curve method, see Hansen (1992). For the zero-th order regularization procedure we plot on a log-log scale the variation of $||\boldsymbol{x}_\lambda||$ against the fitness measure, namely the residual norm $||\mathbb{C}\boldsymbol{x}_\lambda - \boldsymbol{d}||$ for a wide range of values of $\lambda > 0$. In many applications this graph results in a L-shaped curve and the choice of the optimal regularization parameter $\lambda > 0$ is based on selecting approximately the corner of this L-curve.

3. Numerical results

In this paper we investigate the solution of Stokes problem given by equations (2) and (3) in a simple two-dimensional non-smooth geometry, such as the square $\Omega = \{(x, y)| -\frac{1}{2} < x < \frac{1}{2}, -\frac{1}{2} < y < \frac{1}{2}\}$.

In order to investigate the convergence and the stability of the solution obtained by using the technique employed, we consider the following test example, namely the analytical expressions for the three harmonic functions f, g and P are given by:

$$f^{(an)} = 4y^3 - 12x^2y \tag{18}$$

$$g^{(an)} = 4x^3 + 3xy - 1 - 12xy^2 \tag{19}$$

$$P^{(an)} = 24xy - 2x \tag{20}$$

with the corresponding fluid velocity given by:

$$u^{(an)} = -x^2 + 4y^3 \tag{21}$$

$$v^{(an)} = 4x^3 + 2xy - 1 \tag{22}$$

The numerical results obtained for P on the boundary of the square, as a function of ζ - the fraction of the number of boundary elements measured anticlockwise round the boundary from the point $(-\frac{1}{2}, \frac{1}{2})$ when using 20, 40 and 80 constant and discontinuous boundary elements, as well as the analytical solution given by equation (20), are plotted in Figure 1 (a) and (b), respectively.

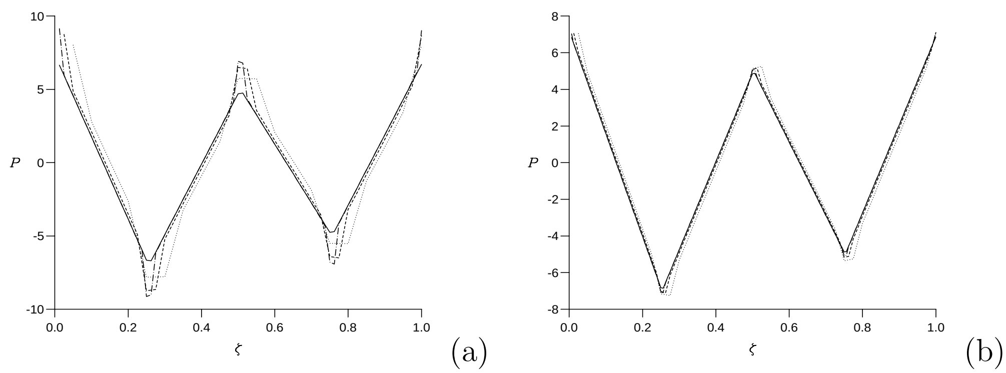
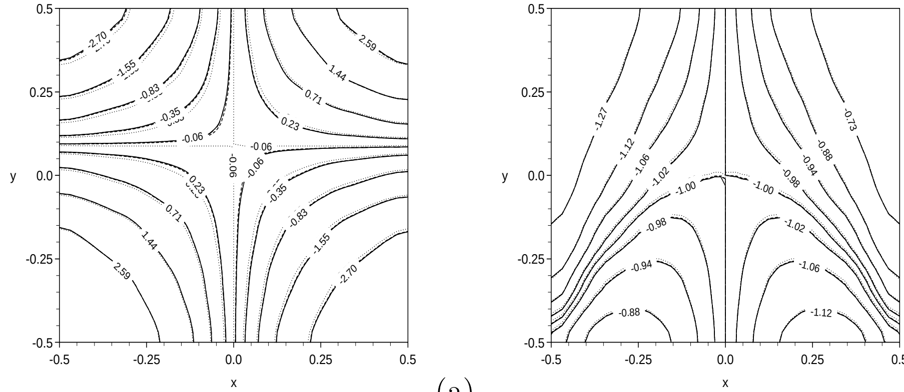


Figure 1: The numerical boundary solution for P obtained with $N = 20(\cdots)$, $N = 40(- \ \ - \ \ -)$ and $N = 80(- \cdot - \cdot -\cdot)$ (a) *constant* and (b) *discontinuous linear* boundary elements, and the analytical solution (——) with u and v specified on the boundary.

From these figures it can be observed that the numerical solutions accurately estimate the analytical solution and converge to their corresponding analytical solution as the number of boundary elements increases. Similar observations can be made for the solutions for f and g on the boundary and therefore these results are not presented. The oscillations which appear at the corners for f, g and P no longer appear for u and v, which is to be expected as the combinations forming them are prescribed boundary conditions. The oscillatory behaviour of the harmonic functions near the corners when using CBEM, diminish greatly when using DLBEM. The numerical solutions for u and v in the solution domain are in excellent agreement with the analytical solutions, e.g. see Figure 2 (b) for v. Moreover, the pressure inside the domain is also accurate in comparison with the analytical solution as N increases, as can be seen in Figure 2 (a).

Solving the direct problem with u and v known on Γ, we obtain the pressure



(a) (b)

Figure 2: The numerically obtained interior (a) pressure P, and (b) fluid velocity v, with $N = 20(\cdots)$, $N = 40(- \quad - \quad -)$ and $N = 80(- \cdot - \cdot -\cdot)$ *discontinuous linear* boundary elements and the analytical solution (——) with u and v specified on the boundary.

P on Γ and, in particular, over Γ^*. This numerically calculated pressure, denoted by $P^{(n)}$ is used in the inverse problem in which v is assumed unknown on Γ_0. The numerical solutions are obtained for the unspecified values of both the normal component of the fluid velocity and the boundary pressure.

Whilst in the direct Stokes problem we observed that the difference between the analytical solution and the numerical results for P using $N = 80$ was less than 1%, in the inverse problem we found $N = 40$ was sufficiently large for the numerical solution to agree graphically with the corresponding numerical solution from the direct problem.

Further, it is to be noted that if for a fixed value of N, instead of its numerical results from solving the direct problem, we impose the analytical value $P^{(an)}$ for the extra condition $P|_{\Gamma^*}$, we would be generating a "numerical noise" which originates from the discretisation of Γ into a finite number of straight line boundary elements and from the piecewise constant function approximation of $P^{(an)}$ over each boundary element. This "numerical noise" is $\boldsymbol{O}(||P^{(an)} - P^{(n)}||)$ which becomes zero when we impose the numerically obtained data $P^{(n)}$ for the extra condition over Γ^*. Consequently, the regularization parameter $\lambda > 0$ is very small (can be chosen as small as the computer double precision, namely 10^{-15}), because in this case we are really using errorless boundary data.

The stability of the regularized boundary element technique is investigated by adding small amounts of random noise into the input boundary data in order to simulate measurements errors which are innately present in a data set of any practical problem. Hence, we perturb the boundary data - data obtained for the direct problem - by adding random noisy perturbations ϵ to the boundary pressure $P^{(n)}$, namely $\tilde{P} = P^{(n)} + \epsilon$. The random error ϵ is generated by using the NAG routine G05DDF and it represents a Gaussian random variable with mean zero and standard deviation σ, which is taken to be some percentage α of the maximum value of $P^{(n)}$, i.e. $\sigma = \max|P^{(n)}| \times \frac{\alpha}{100}$.

Figure 3 shows the L-curve for the inverse problem as a log-log plot of the smoothness measure, namely the norm $||\boldsymbol{x}_\lambda||$ of the solution, against the fitness measure, namely the residual norm $||\mathbb{C}\boldsymbol{x}_\lambda - \boldsymbol{d}||$, for various amounts of noise $\alpha = \{5, 10, 20\}$ introduced in $P^{(n)}|_{\Gamma^*}$ and for the various values of the regularization parameter λ taken from the range $[10^{-13}, 10^{-1}]$. We choose the

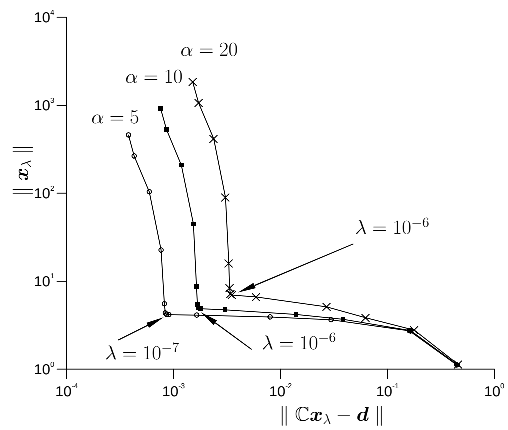
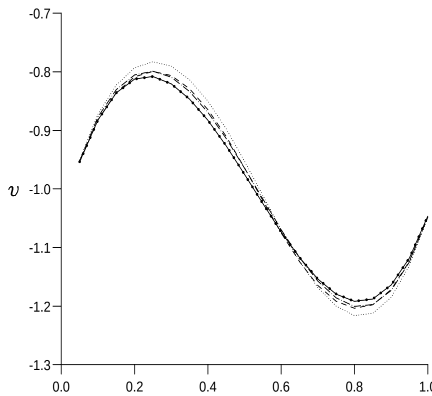


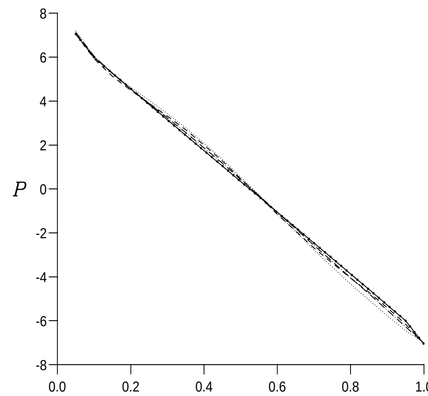
Figure 3: The L-curve plot of the solution norm $\| \boldsymbol{x}_\lambda \|$ as a function of the residual norm $\| \mathbb{C}\boldsymbol{x}_\lambda - \boldsymbol{d} \|$ in the inverse Stokes problem for $N = 40$, $\lambda = [10^{-13}, 10^{-1}]$, when various level of noise $\alpha = \{5, 10, 20\}$ are added.

optimal value of the regularization parameter λ, corresponding to the coner of the L-curve, as $\lambda_{opt} = \boldsymbol{O}(10^{-7})$ if $\alpha = 5$ and $\lambda_{opt} = \boldsymbol{O}(10^{-6})$ if $\alpha = 10, 20$.

The numerical solution for the retrieved normal velocity v and pressure P over Γ_0, obtained using both the exact and noisy data, for λ_{opt} remains stable and agrees with the values specified in the direct problem reasonably well according to the amount of noise introduced in the input data for pressure $P|_{\Gamma^*}$ as can be seen in Figure 4 (a) and (b).



(a)



(b)

Figure 4: The numerical boundary solution for (a) the fluid velocity $v|_{\Gamma_0}$, and (b) the pressure $P|_{\Gamma_0}$ retrieved on the boundary $\Gamma_0 = \Gamma_1$, using $N = 40$ *discontinuous linear* boundary elements when various levels of noise are introduced in $P|_{\Gamma^*=\Gamma_2\cup\Gamma_3\cup\Gamma_4}$, namely, direct solution (——), $\alpha = 0(\bullet\bullet\bullet)$, $\alpha = 5(-\ -\ -)$, $\alpha = 10(-\cdot-\cdot-\cdot)$, and $\alpha = 20(\cdots)$ for the inverse Stokes problem.

The numerical solutions for the pressure P inside the solution domain, obtained using both the exact and noisy data, are displayed in Figure 5 and it can be observed that as the amount of noise decreases then the numerical solutions approximate better the solution obtained in the direct problem (which was found to be accurate in comparison with the analytical solution), whilst at the same time remaining stable.

It is important to note that increasing the under-specified part of the boundary the accuracy of the retrieved values decreases.

4. Conclusions

In this paper a novel technique based on a Laplacian decomposition has been introduced for solving direct and inverse Stokes problems. BEM has been applied to the resulting Laplace equations. The technique has been validated for a benchmark test examples in a square domain. It has been concluded that

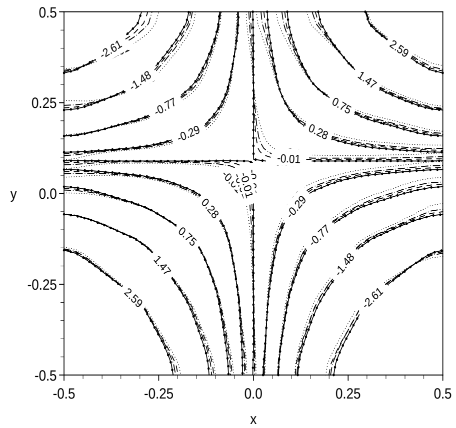


Figure 5: The lines of constant pressure P inside the cavity Ω obtained with $N = 40$ *discontinuous linear* boundary elements when various levels of noise are introduced in $P|_{\Gamma^*=\Gamma_2\cup\Gamma_3\cup\Gamma_4}$, namely, direct solution (——), $\alpha = 0(\bullet\bullet\bullet)$, $\alpha = 5(-\ -\ -)$, $\alpha = 10(-\cdot-\cdot-\cdot)$, and $\alpha = 20(\cdots)$ for the inverse Stokes problem.

the regularized boundary element technique retrieves an accurate, stable and convergent numerical solution, both on the boundary and inside the domain, with respect to increasing the number of boundary elements and decreasing the amount of noise in the input data.

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A. E. Curteanu, L. Elliott, D. B. Ingham & D. Lesnic
Department of Applied Mathematics
University of Leeds
Leeds, LS2 9JT, UK
email:*anca@maths.leeds.ac.uk*