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CURRENTS ON LIE GROUPS

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Abstract. In the first part of the paper are reminded the definitions of currents defined on a differentiable manifold. Let G be a Lie group, $\dim G = n$. Because G is in particular a differentiable manifold, we can define currents on a Lie group. The goal of this paper is to study the properties of currents defined on Lie groups. Some remarks relative to this type of current are pointed in the last part.

1. Introduction

We will consider a differentiable manifold, denote by M, with $\dim M = n$, and $D^p(M)$ the space of differentiable p-forms.

Definition 1. *It says p-current on M an application:*

$$T : \rho \in D^{n-p}(M) \longrightarrow < T, \rho > \in R$$

with:

i) T is linear
ii) T is continuous.

We will denote by $D'^p(M) = \{T : M \longrightarrow \mathbf{R} \text{ where } T \text{ is } p\text{-current}\}$ the set of p-currents defined on M.

Let G be a Lie group, $\dim G = n$. Because G is in particular a differentiable manifold, we can define currents on a Lie group.

Definition 2. *Let $S \in D'^p(G)$, $T \in D'^q(G)$ with $A = Supp(S)$ and $B = Supp(T)$. We say that the currents S and T are convoluable and their convolution is the current:*

$$S * T \overset{def}{=} \gamma(S \otimes T) \in D'^{p+q-n}(G)$$

if for every $\gamma \in D'^{2n-p-q}(G)$ we have:

$$\gamma^{-1}(Supp(\rho)) \cap (A \times B) = compact$$

We can remark that the currents' convolution does not exists always because $\gamma(S \otimes T)$ does not exists always. For example, let $G = \mathbf{R}$, $\rho \in C^{\infty}$, $Supp(\rho) = [-1, 1], T, S \in D'^{1}(\mathbf{R})$ with $Supp(T) = Supp(S) = \mathbf{R}$. Then $S * B$ does not exists because $\gamma^{-1}(Supp(\rho)) \cap (A \times B)$ is not compact.

2. Properties and remarks

We will give now some sufficient conditions for the convolution's existence.

Proposition 1.*If one of the following conditions are true, the currents' convolution always exists:*

a) *G is compact;*
b) *γ is proper;*
c) *one of the currents is with compact support.*

Proof.

The conditions a) and b) are easily proved by the convolution's definition.

For the proof of condition c) we will consider $A = Supp(S)$ - compact, $B = Supp(T)$. By definition $S * T$ exists if and only if for every $\gamma \in D'^{2n-p-q}(G)$, we have $\gamma^{-1}(Supp(\rho)) \cap (A \times B)$ =compact.

Let

A=$\{(\xi, \mu)$

If $(\xi, \mu) \in$ A $\Longrightarrow \xi \in A, \mu \in A^{-1}Supp(\rho)$, so

$$A \subseteq A \times (A^{-1}Supp(\rho)).$$

But because A is compact, A^{-1} is compact, $Supp(\rho)$ is compact, it results A is compact, which is proving our affirmation.

Examples.

1) Let $T \in D'^{p}(G)$, $\delta_a \in D'^{n}(G)$ and L_a the left translation. We have:

$$\delta_a * T = L_a T.$$

2) For $a, b \in G$ we have:

$$\delta_a * \delta_b = \delta_{ab}.$$

Proposition 2.*a) Let $T \in D'^p(G)$, $S \in D'^q(G)$, $R \in D'^r(G)$. Supposing that those currants are convoluable, we have:*

$$S*(T*R)=(S*T)*R.$$

b) Let G a commutative Lie group, $T \in D'^p(G)$, $S \in D'^q(G)$ convoluable currents. Then:

$$S*T=(-1)^{(n-p)(n-q)}\,T*S.$$

Proof.
We consider:

$$I : z \in G \longrightarrow I(z) \stackrel{def}{=} z \in G.$$

Then we have:

$$\gamma \times I : (x,y,z) \in G \times G \times G \longrightarrow (\gamma \times I) : (x,y,z) \stackrel{def}{=} (xy,z) \in G \times G$$

and we can say that:

$$\begin{aligned}
\gamma(\gamma \times I)(S \otimes T \otimes R) &= \gamma(\gamma(S \otimes T) \otimes I(R)) \\
&= \gamma((S*T) \otimes R) \\
&= (S*T)*R \\
\gamma(I \times \gamma)(S \otimes T \otimes R) &= \gamma(I(S) \otimes \gamma(T \otimes R)) \\
&= \gamma(S \otimes (T*R)) \\
&= S*(T*R)
\end{aligned}$$

But:

$$\begin{aligned}
\gamma(\gamma \times I)(x,y,z) &= xyz \\
\gamma(I \times \gamma)(x,y,z) &= xyz
\end{aligned}$$

and so those two applications are the same, obtaining the associativity for the currents convolution.

Proposition 3. *Let $T \in D'^p(G)$, $S \in D'^q(G)$ convoluable currents, Then we have:*

$$Supp(S*T) \subseteq Supp(S)\,Supp(T).$$

Proof.

$Supp(S{*}T) = Supp(\gamma(S{*}T)) \subseteq \gamma(Supp(S{*}T)) = \gamma\left(Supp\left(S\right) \times Supp\left(T\right)\right) = Supp\left(S\right) Supp\left(T\right).$

We can remark then the inclusion from proposition is strictly. Indeed, let G=($\mathbf{R}, +$) and

$$S\left(x\right) \stackrel{def}{=} \left\{ \begin{array}{c} 0 \text{ if } x \in R\backslash\{0\} \\ 1 \text{ if } x = 0 \end{array} \right. ,$$
$$T\left(x\right) \stackrel{def}{=} \left\{ \begin{array}{l} 0 \text{ if } \left|x\right| \geq 1 \\ 1 \text{ if } \left|x\right| < 1 \end{array} \right.$$

Then $\left(S * T\right)\left(x\right) = \int_R S\left(y\right) T\left(x - y\right) dy = 0$. It results that

$$Supp(S * T) = \phi$$

On another hand we have: $Supp\left(S\right) + Supp\left(T\right) = \{x \text{ where } \left|x\right| \leq 1\}$. It results that

$$Supp(S * T) Supp\left(S\right) + Supp\left(T\right).$$

Proposition 4. *Let G, $G\prime$ two Lie groups and $\pi : G \longrightarrow G'$ a morphism of Lie groups. Then for every $T \in D'^p\left(G\right)$, $S \in D'^q\left(G\right)$ convoluable we have*

$$\pi(S * T) = \pi\left(S\right) * \pi\left(T\right).$$

Proof.

We have:

$$< \pi\left(S\right) * \pi\left(T\right), \varphi >=< \pi\left(S\right) \otimes \pi\left(T\right), \varphi\left(xy\right) >$$
$$< (\pi \times \pi)(S \otimes T), \varphi\left(xy\right) >=< (S \otimes T), \pi^* \varphi\left(xy\right) >$$
$$< \mu\left(S * T\right), \varphi >=< S \otimes T, \pi^* \varphi\left(xy\right) >$$

from where it results the equality.

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