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STANDARD MONOMIALS OF SOME SYMMETRIC SETS

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Abstract.We give a new description of the vanishing ideal of some symmetric sets $S \subseteq \{0,1\}^n$over the field of complex numbers. As an application we determine the deglexstandard monomials for S over C. It turns out that the standard monomials can be described in terms of certain generalized ballot sequences. This extends some results obtained in [2] and [6].

Key words: Hilbert function, Radon map, set family, standard monomial, skew tableau.

1.Introduction

Let n be a positive integer and $[n]$ stand for the set $\{1, 2, \ldots, n\}$. The family of all subsets of $[n]$ is denoted by $2^{[n]}$. For an integer $0 \leq t \leq n$ we set

$$S_t = \{w \in \{0,1\}^n ; \text{the Hamming weight of } w \text{ is } t\} .$$

A symmetric set $S \subseteq \{0,1\}^n$ is of the form $S = S_{c_1} \cup \cdots \cup S_{c_k}$, where $0 \leq c_1 < \cdots < c_k \leq n$ are integers. S can be considered as a point set in $\mathbb{F}^n$ for any field $\mathbb{F}$.

As usual, $\mathbb{F}[x_1, \ldots, x_n]$ denotes the ring of polynomials in $x_1, \ldots, x_n$ over $\mathbb{F}$. For a subset $F \subseteq [n]$ we write $x_F = \prod_{j \in F} x_j$. In particular, $x_\emptyset = 1$. Let $v_F \in \{0,1\}^n$ denote the characteristic vector of a set $F \subseteq [n]$. For a family of subsets $\mathcal{F} \subseteq 2^{[n]}$, let $V(\mathcal{F}) = \{v_F \colon F \in \mathcal{F}\} \subseteq \{0,1\}^n \subseteq \mathbb{F}^n$. A polynomial $f \in \mathbb{F}[x_1, \ldots, x_n] = X$ can be considered as a function from $V(\mathcal{F})$ to $\mathbb{F}$ in the straightforward way. We note also that $V(\mathcal{F}) \subseteq \{0,1\}^n$, and conversely, for any $S \subseteq \{0,1\}^n$ there exists an $\mathcal{F} \subseteq 2^{[n]}$ such that $S = V(\mathcal{F})$.

Several interesting results on finite set systems $\mathcal{F} \subseteq 2^{[n]}$ can be naturally formulated as statements concerning *polynomial functions on* $S = V(\mathcal{F})$.

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For instance, certain inclusion matrices can be viewed naturally in this setting. Also, the approach to the complexity of Boolean functions, initiated by Smolensky [10] and developed further by Bernasconi and Egidi [4], leads to such questions.

To study polynomial functions on S, it is natural to consider the ideal $I(S)$:

$$I(S) := \{f \in X \colon f(v) = 0 \text{ whenever } v \in S\}.$$

In fact, substitution gives rise to a $\mathbb{F}$-homomorphism from X to the ring of $\mathbb{F}$-valued functions on S. This homomorphism is seen to be surjective by an easy interpolation argument, and the kernel is exactly $I(S)$. This way one can identify $S/I(S)$ with the space of $\mathbb{F}$-valued functions on S. In particular, $\dim_{\mathbb{F}} S/I(S) = |\mathcal{F}| = |S|$.

2.Grbner bases, standard monomials and Hilbert functions

We recall now some basic facts concerning Gröbner bases and Hilbert functions in polynomial rings. A total order $\prec$ on the monomials (words) composed from variables $x_1, x_2, \ldots, x_n$ is a *term order*, if 1 is the minimal element of $\prec$, and $uw \prec vw$ holds for any monomials u, v, w with $u \prec v$. There are many interesting term orders. For the rest of the paper we assume that the term order $\prec$ we work with is the *deglex* order. Let $u = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ and $v = x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ be two monomials. Then u is smaller than v with respect to deglex ($u \prec v$ in notation) iff either $\deg u < \deg v$, or $\deg u = \deg v$ and $i_k < j_k$ holds for the smallest index k such that $i_k \neq j_k$. Note that we have $x_n \prec x_{n-1} \prec \ldots \prec x_1$.

The *leading monomial* $\mathrm{lm}(f)$ of a nonzero polynomial $f \in X$ is the largest (with respect to $\prec$) monomial which appears with nonzero coefficient in f when written as a linear combination of monomials.

Let I be an ideal of X. A finite subset $G \subseteq I$ is a *Gröbner basis* of I if for every $f \in I$ there exists a $g \in G$ such that $\mathrm{lm}(g)$ divides $\mathrm{lm}(f)$. In other words, the leading monomials of the polynomials from G generate the semigroup ideal of monomials $\{\mathrm{lm}(f) : \ f \in I\}$. Using that $\prec$ is a well founded order, it follows that G is actually a basis of I, i.e. G generates I as an ideal of X. It is a fundamental fact (cf. [11, Chapter 1, Corollary 3.12] or [1, Corollary 1.6.5, Theorem 1.9.1]) that every nonzero ideal I of X has a Gröbner basis.

A monomial $w \in X$ is called a *standard monomial for* I if it is not a leading monomial of any $f \in I$. Let $\mathrm{sm}(I, \mathbb{F})$ stand for the set of all standard monomials of I with respect to the term-order $\prec$ over $\mathbb{F}$. It follows from the definition and existence of Gröbner bases (see [11, Chapter 1, Section 4]) that for a nonzero ideal I the set $\mathrm{sm}(I, \mathbb{F})$ is a basis of the $\mathbb{F}$-vector-space X/I. More precisely every $g \in X$ can be written uniquely as $g = h + f$ where $f \in I$ and h is a unique $\mathbb{F}$-linear combination of monomials from $\mathrm{sm}(I, \mathbb{F})$.

If $S \subseteq \{0,1\}^n$, then $x_i^2 - x_i \in I(S)$, hence x_i^2 is a leading monomial for $I(S)$. It follows that the standard monomials for this ideal are all square-free, i.e. of form x_G for $G \subseteq [n]$. We put

$$\mathrm{Sm}(S, \mathbb{F}) = \{G \subseteq [n] : \ x_G \in \mathrm{sm}(I(S), \mathbb{F})\} \subseteq 2^{[n]}.$$

It is immediate that $\mathrm{Sm}(S, \mathbb{F})$ is a downward closed set system. Also, the standard monomials for $I(S)$ form a basis of the functions from S to $\mathbb{F}$ (see Section 4 in [2]), hence

$$|\mathrm{Sm}(S, \mathbb{F})| = |\mathcal{F}|.$$

It is a fundamental fact that if $\mathcal{G}$ is a Gröbner basis of I, then with $\mathcal{G}$ we can reduce every polynomial into a linear combination of standard monomials for I.

Let I be an ideal of $X = \mathbb{F}[x_1, \ldots, x_n]$. The *Hilbert function* of the algebra X/I is the sequence $h_{X/I}(0), h_{X/I}(1), \ldots$. Here $h_{X/I}(m)$ is the dimension over $\mathbb{F}$ of the factor-space $\mathbb{F}[x_1, \ldots, x_n]_{\leq m}/(I \cap \mathbb{F}[x_1, \ldots, x_n]_{\leq m})$ (see [5, Section 9.3]).

In the case when $I = I(S)$ for some set $S \subseteq \{0,1\}^n$, then the number $h_S(m) := h_{X/I}(m)$ is the dimension of the space of functions from S to $\mathbb{F}$ which can be represented as polynomials of degree at most m. On the other hand, $h_{X/I}(m)$ is the number of standard monomials of degree at most m with respect to an arbitrary degree-compatible term order, for instance deglex.

In this paper we describe the deglex standard monomials for the ideal $I(S)$ where S is a symmetric set such that for each c at most one of the subsets S_c and S_{n-c} is in S (we say that S contains no complementary levels). The main result is Theorem 3.5 which gives a combinatorial description of $\mathrm{sm}(I(S),)$.

As noted by A. Bernasconi and L. Egidi in [4], it would be valuable to describe the (reduced) Gröbner bases of an arbitrary symmetric set. Our result is a step into this direction.

3. PRELIMINARIES

Throughout the paper n is a positive integer. Let m, k be nonnegative integers such that $0 \leq k \leq n - m \leq m$.

Definition 2.1.*A skew tableau t of shape $s = (m, n-m, k)$ is a collection of n boxes (cells) appearing in two rows, there are m boxes in the first row and $n-m$ boxes in the second one. Moreover, the first row is shifted to the right with k boxes. These boxes are filled with the elements of $[n]$, each box contains exactly one integer, and different boxes contain different elements.*

It is easy to see that there are $n!$ tableaux of shape $(m, n-m, k)$.

A skew tableau t of shape $(m, n-m, k)$ is called *standard* if the numbers increase along the rows and down the columns of t.

For example, if $n = 6$, then

$$\begin{array}{|c|c|c|c|c|} \cline{2-5} \multicolumn{1}{c|}{} & 2 & 5 & 3 & 6 \\ \hline 7 & 1 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \tag{1}$$

are two skew tableaux, the first is of shape (4,2,1), the second one is of shape (4,2,0). This latter is also a standard tableau.

The symmetric group Sym_n acts on the set of skew tableaux: for $\pi \in Sym_n$ and an $(m, n-m, k)$ skew tableau t the skew tableau πt is also a $(m, n-m, k)$ skew tableau and it will have $\pi(j)$ in the box where t contains j. Two skew tableaux t and t' associated with the same type $(m, n-m, k)$ are *row (resp. column) equivalent* if t' can be obtained from t by permuting numbers in the same rows (resp. columns). The (row) equivalence classes are called *skew tabloids*. The skew tabloid of a skew tableau t is denoted by $\{t\}$. Following [9], we depict the skew tabloid $\{t\}$ by just erasing the vertical lines from the picture of t. The skew tabloids corresponding to the skew tableaux of (1) may be drawn as

$$\begin{array}{ccccc} \cline{2-5} & 2 & 5 & 3 & 6 \\ \hline 7 & 1 & & & \\ \cline{1-2} \end{array} \quad \text{and} \quad \begin{array}{cccc} \hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 & & \\ \cline{1-2} \end{array}$$

For an arbitrary field $\mathbb{F}$, let $M^{m,k}$ denote the linear space over $\mathbb{F}$ whose basis elements are the tabloids of shape $(m, n-m, k)$, obviously $\dim M^{m,k} = \binom{n}{m}$.

Let t be a skew tableau of shape $(m, n-m, k)$. We denote by e_t the sum in $M^{m,k}$ of skew tabloids

$$e_t := \sum_{\pi \in C(t)} \operatorname{sign}(\pi) \cdot \pi\{t\}, \tag{2}$$

where the summation is for those permutations $\pi \in Sym_n$ which stabilize the columns of t.

Example. Let $n = 6$, $s = (3, 3, 1)$, and

$$t = \begin{array}{|c|c|c|c|} \cline{2-4} \multicolumn{1}{c|}{} & 2 & 5 & 3 \\ \hline 4 & 1 & 6 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}.$$

Then

$$e_t = \frac{\overline{\quad 2\;\;5\;\;3}}{\underline{4\;\;1\;\;6\quad}} - \frac{\overline{\quad 1\;\;5\;\;3}}{\underline{4\;\;2\;\;6\quad}} - \frac{\overline{\quad 2\;\;6\;\;3}}{\underline{4\;\;1\;\;5\quad}} + \frac{\overline{\quad 1\;\;6\;\;3}}{\underline{4\;\;2\;\;5\quad}}. \tag{3}$$

Let Y be the linear space over $\mathbb{F}$ whose basis elements are the $x_H, H \subseteq [n]$. We obtain an inner product on Y by setting

$$\langle x_H, x_K \rangle := \delta_{H,K},\;\; H, K \subseteq [n].$$

Let P^i denote the linear subspace of Y spanned by the x_H, $H \subseteq [n]$, $|H| = i$. Then, for $d \leq k$, the adjoint Radon maps $r^{k,d} : P^d \to P^k$ are defined by

$$r^{k,d}(x_H) := \sum_{G \supseteq H,\;\; |G|=k} x_G. \tag{4}$$

To a skew tableau t of shape $(m, n-m, k)$, we can assign a squarefree monomial in variables $x_1, \ldots, x_n$ of degree $n-m$ in the following way: let $\phi_{n-m}(t)$ denote the squarefree monomial of degree $n-m$ whose indeterminates are indexed with the entries of the second row of t. Please note that the value of $\phi_{n-m}(t)$ depends only on $\{t\}$. We have $\phi_3(t) = x_1x_4x_6$ for t in the preceding example.

It is easy to see that the map ϕ_{n-m} defines a linear map from $M^{m,k}$ to P^{n-m}. Let $p(e_t)$ be the image of the element e_t defined by this linear map.

For the rest of the paper we assume that our base field is the field of complex numbers .

4. The result

Our aim is to describe the (deglex) standard monomials for certain symmetric sets. Since $S \subseteq \{0, 1\}^n$, we have $x_i^2 - x_i \in I(S)$ for every i, hence we may restrict our attention to polynomials involving squarefree monomials only. The ring we work with is

$$Y := [x_1, \ldots, x_n]/\langle x_1^2 - x_1, \ldots, x_n^2 - x_n \rangle.$$

Y is a -vector space of dimension 2^n and it carries a Sym_n-module structure. The squarefree monomials x_K in $x_1, \ldots, x_n$ form a basis of Y over . In particular, we can speak about the degree of elements $f \in Y$: the degree of a squarefree monomial x_K is simply $|K|$. Also we can identify Y with the space Y introduced in the preceding section, and hence may work with the inner product $\langle\ ,\ \rangle$ on Y.

A simple counting argument shows that Y is isomorphic to the $\mathbb{C}$-algebra of all functions from $\{0,1\}^n$ to $\mathbb{C}$. A similar counting shows that the subspace of all functions vanishing on $0,1$-vectors of Hamming weight at most d (where $0 \leq d \leq n$) is spanned by all monomials x_K with $|K| > d$. This in turn implies that if $f \in Y$ and $\deg f = d$, then there exists a $0,1$-vector v of Hamming weight at most d such that $f(v) \neq 0$. Let $J(S)$ denote the image of $I(S)$ in Y.

We recall the main result of [4] which gives the Hilbert function of a symmetric set S over any field of characteristic 0. Let $S = S_{c_1} \cup \cdots \cup S_{c_k}$ be a symmetric subset of $\{0,1\}^n$, where $0 \leq c_1 < \cdots < c_k \leq n$. For a fixed natural number m let us define recursively a function $fam(c)$ on the set $\{c_1, \ldots, c_k\}$. If $c_i \leq m$ then $fam(c_i) := c_i$ else let $fam(c_i)$ be the largest integer r not larger than m such that $r \notin \{fam(c_1), \ldots, fam(c_{i-1})\}$. Let l be the largest index such that $c_l \leq m$.

Theorem 3.1*(A. Bernasconi-L. Egidi)*

$$h_S(m) = \sum_{i=1}^{l} \binom{n}{c_i} + \sum_{i=l+1}^{k} \min\left\{ \binom{n}{c_i}, \binom{n}{fam(c_i)} \right\} \tag{5}$$

In particular, if $S_c \subseteq S$ but $S_{n-c} \not\subseteq S$, then for $S' := (S \setminus S_c) \cup S_{n-c}$, we have $h_S(m) = h_{S'}(m)$.

From now on we assume that S is a symmetric set containing no complementary levels i.e. $S = S_{c_1} \cup \cdots \cup S_{c_k}$, where $0 \leq c_1, \ldots, c_k \leq n$ are pairwise distinct integers and at most one of the subsets S_c and S_{n-c} is in S. Let $d_i := \min(c_i, n - c_i)$. By changing the order of indices we may assume that $0 \leq d_1 < \cdots < d_k \leq \mathbb{F}racn2$.

Corrolary 3.2 *For $j = 1, \ldots, k$ we have*

$$h_S(d_j + k - j) = \sum_{i=1}^{j} \binom{n}{d_i} + \sum_{l=1}^{k-j} \binom{n}{d_j + l}. \tag{6}$$

Proof. By the last statement of Theorem 3.1 we may assume that $c_i = d_i$ for all i. Now the formula follows immediately from the Theorem. ☐

As a consequence of Corollary 3.2, we have

$$\dim J(S)_{\leq d_j+k-j} = \sum_{i=0}^{d_j} \binom{n}{i} - \sum_{l=1}^{j} \binom{n}{d_l}. \tag{7}$$

DEFINITION 3.3. *A (finite) 0-1 sequence is a ballot sequence if in each prefix the number of zeros is not smaller than the number of ones. A (finite) 0-1 sequence is a k-ballot sequence if by putting k zeros in front of the original sequence we get a ballot sequence.*

DEFINITION 3.4. *A (finite) increasing sequence of positive integers is k-ballot if its characteristic sequence is a k-ballot sequence. Similarly, a squarfree monomial is k-ballot if the characteristic sequence of its variables in increasing order is a k-ballot sequence.*

EXAMPLE. The monomial $x_1x_3x_5$ is 1-ballot but it is not (0-)ballot.

REMARK. If a monomial is k-ballot then it is also l-ballot for $l \geq k$.

The main result gives a combinatorical description of the standard monomials for S in terms of shifted ballot sequences.

THEOREM 3.5. *The standard monomials for S of degree not more than $d_1 + k - 1$ are the $(k-1)$-ballots, the standard monomomials for S of degree at least $d_{j-1} + k - j + 2$ and at most $d_j + k - j$ are the $(k-j)$-ballots for $j = 2, \ldots, k$.*

EXAMPLE. Let $n = 6$, $S = S_1 \cup S_4$. The standard monomials for S are: $1; x_1, \ldots, x_6; x_1x_3, \ldots, x_1x_6, x_2x_3, \ldots, x_5x_6$, the 1-ballots of degree at most 2.

The proof consists of three parts:

- we characterize the functions in$Y$ which vanish on S,

- we show that the leading term of such a function canNOT be j-ballot for a certain j,
- by a counting argument we show that the monomials that could be standard according to the above observation are indeed standard monomials.

For the first part, we use two lemmas.

LEMMA 3.6.*Let* $0 \leq c \leq n$ *be an integer,* $0 \neq f \in Y$*,* $\deg f \leq \min(c, n-c) - 1 := d - 1$*. Then the degree of* $g = (\sum x_i - c)f$ *in* Y *is* $\deg f + 1$*.*

Proof. By contradiction: if f is a counterexample, then the "head" of f (the sum of terms of f of maximal degree) is also a counterexample. We assume therefore that f is homogeneous. Recall the discussion at the beginning of the section: there exists a 0-1 vector v of Hamming weight at most $d-1$ such that $f(v) \neq 0$ therefore $g(v) \neq 0$ implying that $g \not\equiv 0$.

As f is a counterexample, we have $\deg g = \deg f$ implying that g is a homogeneous element of degree at most d in Y. The fact that g vanishes on S_c contradicts to Gottlieb's Theorem ([7]) which states that the squarefree monomials of degree $t \leq \min(c, n-m)$ are linearly independent on S_c. □

Iterated application of the Lemma gives the following:

COROLLARY 3.7 *Let* $1 \leq j \leq k$*. Then for* $0 \neq f \in Y$*,* $\deg f \leq d_j - 1$*, the degree of the reduced form of* $(\sum x_i - c_j) \cdot \ldots \cdot (\sum x_i - c_k) \cdot f$ *is* $\deg f + k - j + 1$*.* □

PROPOSITION 3.9 *For the set*

$$H_1 = \left\{ \left(\sum_{i=1}^{n} x_i - c_1 \right) \cdot \ldots \cdot \left(\sum_{i=1}^{n} x_i - c_k \right) \cdot f \mid \ \deg f \leq d_1 - 1 \right\} \subset Y$$

we have $H_1 = J(S)_{\leq d_1 + k - 1}$*.*

Proof. Clearly H_1 is a linear subspace of Y. By Corollary **??** if $0 \neq f \in Y$, $\deg f \leq d_1 - 1$ then $(\sum x_i - c_1) \ldots (\sum x_i - c_k) \cdot f \neq 0$ in Y. We infer that the dimension of H_1 is $\binom{n}{0} + \cdots + \binom{n}{d_1 - 1}$. On the other hand, from (6) we know that $\dim J(S)_{\leq d_1 + k - 1} = \binom{n}{0} + \cdots + \binom{n}{d_1 - 1}$. Using that $H_1 \subseteq J(S)_{\leq d_1 + k - 1}$, and that the dimensions of the two spaces are equal, we are done. □

We can extend the above argument to higher degrees. It follows from (5) that for each monomial ω, with $d_1 + 1 \le \deg \omega \le d_2 - 1$ there exists at least one polynomial $p(\omega) \in J(S_{c_1})$ such that the leading monomial of $p(\omega)$ is ω. Let us choose one $p(\omega)$ for each ω and consider the linear subspace spanned by these $p(\omega)$:

$$L_2 := \langle \{p(\omega) \mid d_1 + 1 \le \deg \omega \le d_2 - 1\} \rangle.$$

With the aid of L_2, we define $H_2 \subseteq Y$ as

$$H_2 := \left\{ \left(\sum_{i=1}^{n} x_i - c_2 \right) \cdot \ldots \cdot \left(\sum_{i=1}^{n} x_i - c_k \right) \cdot f \mid f \in L_2 \right\}.$$

H_2 is clearly a linear space and $H_2 \subseteq J(S)_{\le d_2+k-2}$. Corollary 3.7 shows that the degree (in Y) of any element of H_2 is at least $d_1 + k$ and at most $d_2 + k - 2$ (for $0 \ne f \in L_2$). These imply that $H_1 \cap H_2 = \{0\}$ and the dimension of H_2 is $\binom{n}{d_1+1} + \cdots + \binom{n}{d_2-1}$. By (7) we have $\dim J(S)_{\le d_2+k-2} = \binom{n}{0} + \cdots + \binom{n}{d_1-1} + \cdots + \binom{n}{d_1+1} + \cdots + \binom{n}{d_2-1}$, hence $\dim H_1 + \dim H_2 = \dim J(S)_{\le d_2+k-2}$. We infer that $J(S)_{\le d_2+k-2} = H_1 \oplus H_2$.

Similarly, for $2 \le j \le k$ and for each monomial ω, with $d_{j-1} + 1 \le \deg \omega \le d_j - 1$, there exists at least one polynomial $p(\omega) \in J(S_{c_1} \cup \cdots \cup S_{c_{j-1}})$ such that the leading monomial of $p(\omega)$ is ω. Let us choose one $p(\omega)$ for each ω, and set

$$L_j := \langle \{p(\omega) \mid d_{j-1} + 1 \le \deg \omega \le d_j - 1\} \rangle.$$

Now H_j is defined by

$$H_j := \left\{ \left(\sum_{i=1}^{n} x_i - c_j \right) \cdot \ldots \cdot \left(\sum_{i=1}^{n} x_i - c_k \right) \cdot f \mid f \in L_j \right\}.$$

$H_1, \ldots, H_j$ are subspaces of $J(S)_{\le d_j+k-j}$ and from Corollary 3.7 the degree of any element of H_j is at least $d_{j-1} + k - j + 2$ and at most $d_j + k - j$ (for $f \ne 0$). Therefore the dimension of H_j is $\binom{n}{d_{j-1}+1} + \cdots + \binom{n}{d_j-1}$, and the sum of $H_1, \ldots, H_j$ is a direct sum, and again (7) implies that $\dim J(S)_{\le d_j+k-j} = \dim H_1 + \cdots + \dim H_j$ and hence $J(S)_{\le d_j+k-j} = H_1 \oplus \cdots \oplus H_j$. $\square$

We record the main properties of the subspaces H_j in the statement below:

Lemma 3.9.

1. *We have* $J(S)_{\le d_j+k-j} = H_1 \oplus \cdots \oplus H_j$, *for* $j = 1, \ldots, k$.

2. The nonzero elements of H_1 have degree at most $d_1 + k - 1$.
3. For $2 \leq j \leq k$ the nonzero elements of H_j have degree at least $d_{j-1}+k-j+2$ and at most $d_j + k - j$. □

Proposition 3.10 *Let $g = (\sum x_i - c_1) \cdot \ldots \cdot (\sum x_i - c_k) \cdot f$, where $f \in Y$, $\deg f \leq d_1 - 1$. Then the (deglex) largest monomial of g (in Y) is NOT a $(k-1)$-ballot.*

Before proving Proposition 3.10, we recall that that an $(m, n-m, k)$ skew tableau t has two rows, the first row is shifted to the right with k boxes and has m boxes, the second row has $n-m$ boxes ($m \geq n-m \geq k \geq 0$), and e_t denotes the signed sum of skew tabloids defined in (2).

Remark.It can easily be seen that if t is an $(m, n-m, k)$ *standard* skew tableau (increasing numbers along the rows and down the columns) then the second row of t is a k-ballot sequence. Conversely, from a k-ballot sequence α of integers from $[n]$ and of length $n-m$ one can easily obtain a standard skew tableau of shape $(m, n-m, k)$ whose second row is α.

Let $p(e_t)$ denote the squarefree polynomial corresponding to e_t. Recall that for $k \leq d$, $r^{k,d} : P^d \to P^k$ is the adjoint Radon map. To prove Proposition 3.10, we employ two lemmas.

Lemma 3.11 *Let $x_{i_1} \ldots x_{i_l}$ be an arbitrary squarefree monomial, where $l < n-m-k$ and let t be an $(m, n-m, k)$ skew tableau. Then $r^{n-m,l}(x_{i_1} \ldots x_{i_l}) \perp p(e_t)$.*

Proof. By $l < n-m-k$ there exists a column of t with two elements j_1, j_2 such that $\{j_1, j_2\} \cap \{i_1, \ldots, i_l\} = \emptyset$. Consider now the set M of monomials which appear in $p(e_t)$ and divisible by $x_{i_1} x_{i_2} \cdots x_{i_l}$. (note that M may be empty). The elements of M can be partitioned into pairs. In such a pair of monomials (m_1, m_2) exactly one of the m_i is divisible by x_{j_1} and the other by x_{j_2}, moreover m_1 and m_2 have opposite signs in $p(e_t)$. This implies that the sum of the coefficients in $p(e_t)$ of the monomials of M will be zero, proving the claim. □

Lemma 3.12. *Let t be an $(m, n-m, k)$ skew* standart *tableau. Then the (deglex) smallest monomial of $p(e_t)$ is $\phi_{n-m}(t)$.*

Proof. This is just an easy consequence of the definition of e_t. Since t is standard, the numbers are increasing down the columns. Thus, for each $\sigma \in C(t)$, $\phi_{n-m}(t) \prec \phi_{n-m}(\sigma t)$. □

Proof of Proposition 3.10. In the argument below we work in Y. In particular the monomials considered are the squarefree monomials that appear in the defining basis of Y. Suppose for contradiction that the deglex leading monomial ω of $g = (\sum x_i - c_1) \dots (\sum x_i - c_k) \cdot f$ is not a $(k-1)$-ballot. By Corollary 3.7 we have $\deg \omega = \deg f + k$, and ω is the leading monomial of

$$\sigma_k \cdot \bar{f} = \sum_{1 \le i_1 < \dots < i_k \le n} x_{i_1} \dots x_{i_k} \cdot \bar{f},$$

where $\bar{f}$ is the homogeneous part of top degree in f. These imply also that ω is the deglex leading monomial of $h = r^{\deg f + k, \deg f}(\bar{f})$. Let t be a skew tableau of shape $s = (n - (\deg f + k), \deg f + k, k-1)$. Then by Lemma 3.11 we have $h \perp p(e_t)$.

Let t' be the standard tableau of shape s defined by ω. Note that from Lemma 3.12, the deglex smallest monomial of $p(e_{t'})$ is ω.

Now using that $h \perp p(e_{t'})$, and that ω is a monomial in common in h and $p(e_{t'})$, we obtain that they must share another monomial ω'. By the preceding remark we have $\omega' \succ \omega$. This, however, contradicts to the fact that ω is the largest monomial of h. This completes the proof. □

By applying this result in turn for $S^2 = S_{c_2} \cup \dots \cup S_{c_k}, g_2 \in H_2, \dots, S^k = S_{c_k}, g_k \in H_k$ and using Lemma 3.9, we can complete the proof of Theorem 3.5. Indeed, assume first that $2 \le j \le k$. Then from Proposition 3.10 we know that the set of $(k-j)$-ballot monomials (of degree at least $d_{j-1} + k - j + 2$ and at most $d_j + k - j$) is a subset of the set of standard monomials of the same degree. From (6) we know that

$$h_S(d_j + k - j) = \sum_{i=1}^{j} \binom{n}{d_i} + \sum_{i=1}^{k-j} \binom{n}{d_j + i}$$

and

$$h_S(d_{j-1} + k - j + 1) = \sum_{i=1}^{j-1} \binom{n}{d_i} + \sum_{i=1}^{k-j+1} \binom{n}{d_{j-1} + i}$$

therefore the number of standard monomials of degree at least $d_{j-1}+k-j+2$ and at most $d_j + k - j$ is $\binom{n}{d_j+k-j} + \dots + \binom{n}{d_j} - \binom{n}{d_{j-1}+k-j+1} - \dots - \binom{n}{d_{j-1}+1}$. Thus, it suffices to show that the number of $(k-j)$-ballot monomials in these degrees is the same. This is provided by the Proposition 3.13. A similar reasoning gives the statement for the standard monomials of degree at most $d_1 + k - 1$.

PROPOSITION 3.13 *Let k, l, n be positive integers, $0 \leq k, l \leq n$. The number of k-ballot monomials in $x_1, \ldots, x_n$ of degree not larger than l is $\binom{n}{l} + \binom{n}{l-1} + \cdots + \binom{n}{l-k}$.*

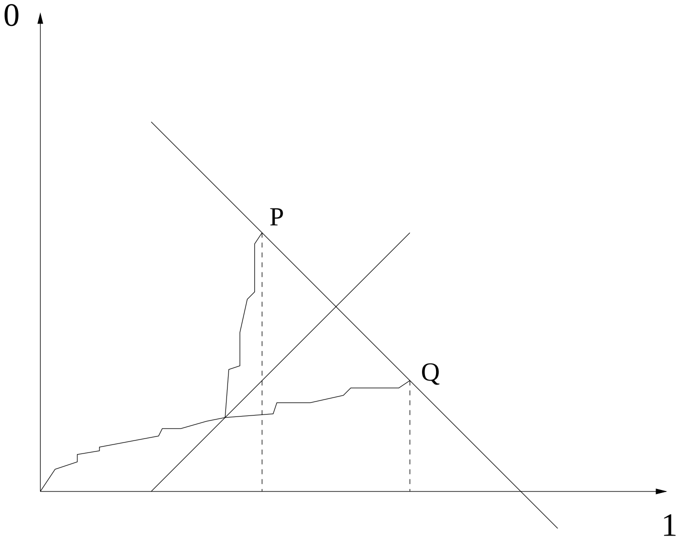

Figure 1:

Proof. From each 0-1 sequence we can construct a lattice path starting at the origin and ending on the line $x+y=n$ in the following manner: we step to the right (draw a horizontal unit segment) for each one and step upwards (draw a vertical unit segment) for each zero. It is easy to see that a 0-1 sequence of length n is a k-ballot sequence iff the appropriate lattice path reaches the line $x+y=n$ without touching the line $y = x - k - 1$ before (Figure 1).

The number of 0-1 sequences (of length n) with l ones is $\binom{n}{l}$. There is a bijection between the "bad" paths (those which reach the line $y = x - k - 1$ before arriving to $(l, n-l)$) and the 0-1 sequences reaching Q. The number of the latter paths is $\binom{n}{l-k-1}$, hence the number of k-ballot sequences with exactly l ones is $\binom{n}{l} - \binom{n}{l-k-1}$, therefore the number of k-ballots containing at most l ones is $\binom{n}{l} + \binom{n}{l-1} + \cdots + \binom{n}{l-k}$. $\square$

5. Concluding remarks

Here we considered sets S which do not contain complementary levels (both S_c and S_{n-c} for some c). Our approach for describing the standard monomials involved three main steps:

1. A description of the ideal $J(S) \subset Y$.
2. A description of the functions in the orthogonal complement of $J(S)$ in Y.
3. A characterization of the deglex-smallest monomials of the elements in the orthogonal complement.

For a general symmetric S the first two steps are feasible but the third one appears to be problematic. It would likely be useful to settle first the case of $S = S_c \cup S_{n-c}$ involving two complementary levels only. We have partial results in this direction.

We add also that if there are complementary levels in S but not separated then the ideas of Theorem 3.5 work. To be more precise, the standard monomials for S have a very similar description, provided we know that for $0 \leq c \leq \mathbb{F}racn2$, if $S_c, S_{n-c} \subseteq S$, then $S_k \subseteq S$ for all integers $c \leq k \leq n-c$.

As we mentioned in the introduction, the most interesting task in this circle of problems is to describe a Grbner basis for $I(S)$ (with respect to a degree compatible order). This is available for example for sets of the form S_c, or, slightly more generally, for $S_c \cup S_{c+1} \cup \cdots \cup S_{c+\ell}$ (cf.[6]).

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