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SOME RESULTS CONCERNING THE NUMBER OF CRITICAL POINTS OF A SMOOTH MAP

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Abstract. In this paper are presented some new results concerning the minimal number of critical points for a smooth map between two manifolds of small codimensions.

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1. Introduction

Let M^{n}, N^{n} be smooth manifolds and let $f : M \to N$ be a smooth mapping. If $x \in M$ consider the rank of f at x to be defined by the non-negative integer

$$\text{rank}_x(f) = \text{rank}(Tf)_x = \dim_{\mathbb{R}}\text{Im}(Tf)_x,$$

where $(Tf)_x : T_x(M) \to T_{f(x)}(M)$ is tangent map of f at x. A point $x \in M$ with the property that $\text{rank}_x(f) = \min(m, n)$ is called *a regular point of f*. Otherwise, the point x is *a critical point* (or a singular point) of f, i.e., x is called a critical point of f if the inequality $\text{rank}_x(f) \leq \min\ (m, n)$ is satisfied. The *critical set* of mapping f is defined by

$$C(f) = \{x \in M \mid x \text{ is a critical point of } f\},$$

and the *bifurcation set* is defined by

$$B(f) = f(C(f))$$

and represents the set of critical values of the mapping f.

Let $\mu(f)$ be the total number of critical points of f, i.e., $\mu(f) = |C(f)|$ (the cardinal number of critical set $C(f)$ defined above).

The φ- *category* of pair (M, N) (or the *functional category* of pair (M, N)) is defined by:

$$\varphi(\text{M, N}) = \min\ \{\mu(f) : f \in C^{\infty}(M, N)\}.$$

It is clear that $0 \leq \varphi(M,N) \leq +\infty$. The relation $\varphi(M,N) = 0$ holds if and only if there is an immersion $M \to N$ $(m < n)$, a submersion $M \to N$ $(m > n)$ or a locally diffeomorfism in any point of M $(m = n)$. (M,N) can be considered a differential invariant of pair (M,N).

Most of the previously known results consist of sufficient conditions on M and N ensuring that $\varphi(M,N)$ is infinite. We are also interested to point out some situations when $\varphi(M,N)$ is finite.

2. $\varphi(M,N)$ for a pair of surfaces

In this paper we review some recent results concerning the invariant $\varphi(M,N)$ in case when manifolds M and N are oriented surfaces. These result are obtained by D. Andrica and L. Funar in papers [2] and [3]. Let us note by $\sum_g$ the oriented surface of genus g and Euler characteristic χ, and by S^2 the 2-dimensional sphere. Denote, also, by [u] the greatest integer not exceeding u. We have:

Theorem 2.1 *Let $\sum$ and $\sum'$ be closed oriented surfaces of Euler characteristics χ and χ', respectively.*

(1) If $\chi' > \chi$, then $\varphi(\sum', \sum) = \infty$;

(2) If $\chi' \leq 0$, then $\varphi(\sum', S^2) = 3$;

(3) If $\chi' \leq -2$, then $\varphi(\sum', \sum_1) = 1$;

(4) If $2 + 2\chi \leq \chi' < \chi \leq -2$, then $\varphi(\sum', \sum) = \infty$;

(5) If $0 \leq |\chi| \leq \dfrac{|\chi'|}{2}$, write $|\chi'| = a\,|\chi| + b$ with $0 \leq b < |\chi|$; then

$$\varphi(\textstyle\sum', \sum) = \left[\dfrac{b}{a-1}\right].$$

In particular, if $g' \geq 2(g-1)^2$, then

$$\varphi(\textstyle\sum_{g'}, \sum_g) = \begin{cases} 0 \text{ if } \dfrac{g'-1}{g-1} \in \mathbb{Z}_+ \\ 1 \text{ otherwise }. \end{cases}$$

The method of proof uses a result given by S. J. Patterson [14]; he gave necessary and sufficient conditions for the existence of a covering of a surface with prescribed degree and ramification orders:

More precisely, let X be a Riemann surface of genus $g \geq 1$, and let $p_1, ..., p_k$ be distinct points of X and $m_1, ..., m_k$ be strictly positive integers so that

$$\sum_{i=1}^{k}(m_i - 1) = 0(\text{mod } 2)$$

and let d be an integer such that $d \geq \max_{i=1,...,k} m_i$. Then there exists a Riemannian surface Y and a holomorphic covering map $f : Y \to X$ of degree d such that there exist k points $q_1, ..., q_k$ in Y so that $f(q_j) = p_j$, and f is ramified to order m_j at q_j and is unramified outside the set $\{q_1, ..., q_k\}$.

Proof of Theorem 2.1.

The first claim is obvious.

For the second affirmation, $\varphi(\sum', S^2) \leq 3$, because any surface is a covering of the 2- sphere branched at three points (from [1]). On the other hand, assume that $f : \sum' \to S^2$ is a ramified covering with at most two critical points. Then f induces a covering map $\sum'$- f^{-1}(B (f))→ S^2 - B (f), where B(f) is the set of critical values and its cardinality $|B(f)| \leq 2$. Therefore one has an injective homomorphism $\pi_1(\sum' - f^{-1}(B(f))) \to \pi_1(S^2 - B(f))$. Now $\pi_1(\sum')$ is a quotient of $\pi_1(\sum' - f^{-1}(B(f)))$ and $\pi_1(S^2 - B(f))$ is either trivial or infinite cyclic, which implies that $\sum' = S^2$.

Next, the unramified coverings of tori are tori; thus any smooth map $f : \sum_{g'} \to \sum_1$ with finitely many critical points must be ramified, so that $\varphi(\sum_{g'}, \sum_1) \geq 1$, if $g' \geq 2$. On the other hand, by Patterson's theorem, there exists a covering $\sum' \to \sum_1$ of degree $d = 2g' - 1$ of the torus, with a single ramification point of multiplicity 2g′-1. From the Hurwitz formula, it follows that $\sum'$ has genus g' , which shows that $\varphi(\sum_{g'}, \sum_1) = 1$.

For the 4th affirmation we need the following auxiliary result:

Lemma 2.1. *$\varphi(\sum', \sum\ )$ is the smallest integer k which satisfies*

$$\left[\frac{\chi' - k}{\chi - k}\right] \leq \frac{\chi' + k}{\chi}.$$

The proof of lemma 2.1 is given in [2] (see also [8]).

Now, assume that $2 + 2\chi \leq \chi' < \chi \leq$-2. If $f : \sum' \to \sum$ was a ramified covering, then we would have $\frac{\chi' + k}{\chi} < 2$, and Lemma 2.1 would imply that $\chi' = \chi$, which is a contradiction. Therefore $\varphi(\sum', \sum\ ) = \infty$ holds.

Finally, assume that $\frac{\chi'}{2} \leq \chi \leq -2$. One has to compute the minimal k satisfying

$$\left[\frac{a\chi - b - k}{\chi - k}\right] \leq \frac{a\chi - b + k}{\chi},$$

or, equivalently,

$$\left[\frac{b + (1-a)k}{\chi - k}\right] \geq \frac{b - k}{\chi}.$$

The smallest k for which the quantity in the brackets is non-positive is $k = \left[\frac{b}{a-1}\right]$, in which case

$$\left[\frac{b + (1-a)k}{\chi - k}\right] \geq 0 \geq \frac{b - k}{\chi}.$$

For k smaller than this value, one has a strictly positive integer on the left-hand side, which is therefore at least 1. However, the right hand side is strictly smaller than 1; hence the inequality cannot hold. This proves the claim.

3. Some results in dimension ≥ 3

The situation changes completely in dimensions $n \geq 3$. The following result is proved in [2].

Theorem 3.1. *Assume that M^n and N^n are compact manifolds. If $\varphi(M^n,N^n)$ is finite and $n \geq 3$, then $\varphi(M^n,N^n) \in \{0, 1\}$. Moreover, $\varphi(M^n, N^n) = 1$ if and only if M^n is the connected sum of a finite covering $\tilde{N}^n$ of N^n with an exotic sphere and M^n is not a covering of N^n.*

Proof.

There exists a smooth map $f : M^n \to N^n$ which is a local diffeomorphism on the preimage of the complement of a finite subset of points. Notice that f is a proper map.

Let $p \in M^n$ be a critical point and let $q = f(p)$. Let $B \subset N$ be a closed ball intersecting the set of critical values of f only at q. We suppose moreover that q is an interior point of B. Denote by U the connected component of $f^{-1}(B)$ which contains p. As f is proper, its restriction to $f^{-1}(B - \{q\})$ is also proper. As it is a local diffeomorphism onto $B - \{q\}$, it is a covering, which implies that

$f : U - f^{-1}(q) \to B - \{q\}$ is also a covering. However, f has only finitely many critical points in U, which shows that $f^{-1}(q)$ is discrete outside this finite set, and so $f^{-1}(q)$ is countable. This shows that $U - f^{-1}(q)$ is connected. As $B - \{q\}$ is simply connected, we see that $f : U - f^{-1}(q) \to B - \{q\}$ is a diffeomorphism. This shows that $f^{-1}(q) \cap U = \{p\}$, for otherwise $H_{n-1}(U - f^{-1}(q))$ would not be free cyclic. Thus $f : U - \{p\} \to B - \{q\}$ is a diffeomorphism. An alternative way is to observe that $f|_{U-\{p\}}$ is a proper submersion because f is injective in a neighborhood of p (except possibly at p). This implies that $f : U - \{p\} \to B - \{q\}$ is a covering and hence a diffeomorphism since $B - \{q\}$ is simply connected.

One can then verify easily that the inverse of $f|_U : U \to B$ is continuous at q; hence it is a homeomorphism. In particular, U is homeomorphic to a ball. Since ∂U is a sphere, the results of Smale imply that U is diffeomorphic to the ball for $n \neq 4$.

We obtain that f is a local homeomorphism and hence topologically a covering map. Thus M^n is homeomorphic to a covering of N^n. Let us show now that one can modify M^n by taking the connected sum with an exotic sphere in order to get a smooth covering of N^n.

By gluing a disk to U, using an identification $h : \partial U \to \partial B = S^{n-1}$, we obtain a homotopy sphere (possibly exotic) $\sum_1 = U \cup_h B^n$. Set $M_0 = M - int(U)$, $N_0 = N - int(B)$. Given the diffeomorphisms $\alpha : S^{n-1} \to \partial U$ and $\beta : S^{n-1} \to \partial B$, one can form the manifolds

$$M(\alpha) = M_0 \bigcup_{\alpha : S^{n-1} \to \partial U} B^n, N(\beta) = N_0 \bigcup_{\beta : S^{n-1} \to \partial B} B^n.$$

Set h=$f|_{\partial U} : \partial U \to \partial B = S^{n-1}$. A map $F : M(\alpha) \to N(h \circ \alpha)$ is then given by

$$F(x) = \begin{cases} x \text{ if x} \in D^n \\ f(x) \text{ if x} \in M_0 \end{cases}.$$

The map F has the same critical points as $f|_{M_0}$; hence it has precisely one critical point less than $f : M \to N$.

We choose $\alpha = h^{-1}$ and we remark that $M = M(h^{-1}) \# \sum_1$, where the equality sign stands for diffeomorphism equivalence. Denote $M_1 = M(h^{-1})$. We obtained above that $f : M = M_1 \# \sum_1 \to N$ decomposes as follows. The restriction of f to M_0 extends to M_1 without introducing extra critical points, while the restriction to the homotopy ball corresponding to the holed $\sum_1$ has precisely one critical point.

Thus, iterating this procedure, one finds that there exist possibly exotic spheres $\sum_i$ so that $f : M = M_k \# \sum_1 \# \sum_2 ... \# \sum_k \to N$ decomposes as follows: the restriction of f to the k - holed M has no critical points, and it extends to M_k without introducing any further critical point. Each critical point of f corresponds to a (holed) exotic $\sum_i$. In particular, M_k is a smooth covering of N.

Now the connected sum $\sum = \sum_1 \# \sum_2 ... \# \sum_k$ is also an exotic sphere. Let $\Delta = \sum - int(B^n)$ be the homotopy ball obtained by removing an open ball from $\sum$. We claim that there exists a smooth map $\Delta \to B^n$ that extends any given diffeomorphism of the boundary and has exactly one critical point. Then one builds up a smooth map $M_k \# \sum \to N$ having precisely one critical point, by putting together the obvious covering on the 1 - holed M_k and $\Delta \to B^n$. This will show that $\varphi(M, N) \leq 1$.

The claim follows easily from the following two remarks. First, the homotopy ball Δ is diffeomorphic to the standard ball by [17], when $n \neq 4$. Further, any diffeomorphism $\varphi : S^{n-1} \to S^{n-1}$ extends to a smooth homeomorphism with one critical point $\Phi : B^n \to B^n$, for example

$$\Phi(z) = \exp\left(-\frac{1}{\|z\|^2}\right)\varphi\left(\frac{z}{\|z\|}\right).$$

For $n = 4$, we need an extra argument. Each homotopy ball $\Delta_i^4 = \sum_i - int(B^4)$ is the preimage $f^{-1}(B)$ of a standard ball B. Since f is proper, we can choose B small enough such that Δ_i^4 is contained in a standard 4-ball. Therefore Δ^4 can be engulfed in S^4. Moreover, Δ^4 is the closure of one connected component of the complement of $\partial\Delta^4 = S^3$ in S^4. The result of Huebsch and Morse from [12] states that any diffeomorphism $S^3 \to S^3$ has a Schoenflies extension to a homeomorphism $\Delta^4 \to B^4$ which is a diffeomorphism everywhere except for one (critical) point. This proves the claim.

Remark finally that $\varphi(M^n, N^n) = 0$ if and only if M^n is a covering of N^n. Therefore if M^n is diffeomorphic to the connected sum $\widetilde{N}^n \# \sum^n$ of a covering $\widetilde{N}^n$ with an exotic sphere $\sum^n$, and if it is not diffeomorphic to a covering of N^n, then $\varphi(M^n, N^n) \neq 0$. Now drill a small hole in $\widetilde{N}^n$ and glue (differently) an n-disk B^n (respectively a homotopy 4-ball if $n = 4$) in order to get $\widetilde{N}^n \# \sum^n$. The restriction of the covering $\widetilde{N}^n \to N^n$ to the boundary of the hole extends (by the previous argument) to a smooth homeomorphism with one critical point over $\sum^n$. Thus $\varphi(M^n, N^n) = 1$.

In the case of small nonzero codimensions we can state the following result (see [2] and [8]):

THEOREM 3.2.*If* $\varphi(M^m, N^n)$ *is finite and either* $m = n+1 \neq 4$, $m = n+2 \neq 4$, *or* $m = n+3 \notin \{5, 6, 8\}$ *(when one assume that the Poincaré conjecture to be true) then M is homeomorphic to a fibration of base N. In particular if* $m = 3, n = 2$ *then* $\varphi(M^3, N^2) \in \{0, \infty\}$, *except possible for* M^3 *a non-trivial homotopy sphere and* $N^2 = S^2$.

In arbitrary codimension we have:

THEOREM 3.3.*Assume that there exists a topological submersion* $f : M^m \to N^n$ *with finitely many critical points, and* $m > n \geq 2$. *Then* $\varphi(M, N) \in \{0, 1\}$ *and it equals 1 precisely when M is diffeomorphic to the connected sum of a fibration* $\widetilde{N}$ *(over N) with an exotic sphere without being a fibration itself.*

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