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ON A CLASS OF SEQUENCES DEFINED BY USING RIEMANN INTEGRAL

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Abstract.The main result shows that if $f : [1, +\infty) \to \mathbf{R}$ is a continuous function such that $\lim\limits_{x\to\infty} xf(x)$ exists and it is finite, then

$$\lim_{n\to\infty} n \int_1^a f(x^n)dx = \int_1^{+\infty} \frac{f(x)}{x} dx,$$

for any $a > 1$. Two applications are given.

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1.Introduction

There are many important classes of sequences defined by using Riemann integral. We mention here only one which is called the Riemann-Lebesgue Lemma: *Let $f : [a, b] \to \mathbf{R}$ be a continuous function, where $0 \le a < b$. Suppose the function $g : [0, \infty) \to \mathbf{R}$ to be continuous and T-periodic. Then*

$$\lim_{n\to\infty} \int_a^b f(x)g(nx)dx = \frac{1}{T} \int_0^T g(x)dx \int_a^b f(x)dx. \tag{1}$$

For the proof we refer to [4] (in special case $a = 0$, $b = T$) and [5]. In the paper [1] we proved that a similar relation as (1) holds for all continuous and bounded functions $g : [0, \infty) \to \mathbf{R}$ of finite Cesaro mean.

In this note we investigate another class of such sequences, i.e. defined by $n \int_1^a f(x^n)dx$, where $f : [1, +\infty) \to \mathbf{R}$ is a continuous function and $a > 1$ is a fixed real number.

2.The main result

Our main result is the following.

Theorem.*Let $f : [1, +\infty) \to \mathbf{R}$ be a continuous function such that $\lim\limits_{x\to\infty} xf(x)$ exists and it is finite. Then, the improper integral $\int_1^\infty \frac{f(x)}{x} dx$ is convergent and*

$$\lim_{n\to\infty} n\int_1^a f(x^n)dx = \int_1^\infty \frac{f(x)}{x}dx, \tag{2}$$

for any $a > 1$.

Proof. Consider $\lim\limits_{x\to\infty} xf(x) = l$, where $l \in \mathbf{R}$. Then, we can find a real number $x_0 > 1$ such that for any $x \geq x_0$ we have

$$\frac{l-1}{x^2} \leq \frac{f(x)}{x} \leq \frac{l+1}{x^2}.$$

Let us choose a real number $m > 0$ satisfying the inequality $l - 1 + m \geq 0$. Then, for any $x \geq x_0$ we have

$$0 \leq \frac{l-1+m}{x^2} \leq \frac{f(x)}{x} + \frac{m}{x^2} \leq \frac{l+1+m}{x^2} \tag{3}$$

Define the function $J : [1, +\infty) \to \mathbf{R}$ by

$$J(t) = \int_1^t \left(\frac{f(x)}{x} + \frac{m}{x^2}\right) dx.$$

The function J is differentiable and we have

$$J'(t) = \frac{f(t)}{t} + \frac{m}{t^2} \geq 0$$

for any $t \geq x_0$. Therefore J is an increasing function on interval $[x_0, +\infty)$. Moreover, by using the last inequality in (3) we get by integration

$$\begin{aligned} J(t) &= \textstyle\int_1^{x_0} \left(\frac{f(x)}{x} + \frac{m}{x^2}\right) dx + \int_{x_0}^t \left(\frac{f(x)}{x} + \frac{m}{x^2}\right) dx \\ &\leq \textstyle\int_1^{x_0} \left(\frac{f(x)}{x} + \frac{m}{x^2}\right) dx + (l+1+m)\int_{x_0}^t \frac{dx}{x^2} \\ &\leq \textstyle\int_1^{x_0} \left(\frac{f(x)}{x} + \frac{m}{x^2}\right) dx + \frac{l+1+m}{x_0}, \end{aligned}$$

for any $t \geq x_0$. It follows that $\lim\limits_{t\to\infty} J(t)$ is finite. But, we have

$$J(t) = \int_1^t \frac{f(x)}{x}dx + m\left(1 - \frac{1}{t}\right),$$

hence
$$\lim_{t\to\infty}\int_1^t \frac{f(x)}{x}dx = \lim_{t\to\infty} J(t) - m,$$
which is finite.

For a fixed real number $a > 1$, denote
$$J(t) = t\int_1^a f(x^t)dx \text{and} U(t) = \int_1^{a^t} \frac{f(x)}{x}dx.$$

Because function $g : [1, +\infty) \to \mathbf{R}$, $g(x) = xf(x)$, is continuous and $\lim\limits_{x\to\infty} g(x)$ is finite, it follows that g is bounded, i.e. we can find $M > 0$ with the property
$$|g(x)| \le M,\ x \in [1, \infty) \tag{4}$$

Changing the variable x by $x = u^t$, we get $dx = tu^{t-1}du$, hence
$$U(t) = t\int_1^a \frac{f(u^t)}{u}du \tag{5}$$

From (4) and (5) we obtain
$$\begin{array}{c} |J(t) - U(t)| = t\left|\int_1^a f(x^t)dx - \int_1^a \frac{f(x^t)}{x}dx\right| = \\ = t\left|\int_1^a \left(f(x^t) - \frac{f(x^t)}{x}\right)dx\right| \le t\int_1^a |f(x^t)|\frac{x-1}{x}dx = \\ = t\int_1^a x^t|f(x^t)|\frac{x-1}{x^{t+1}}dx \le tM\int_1^a \frac{x-1}{x^{t+1}}dx = \\ = Mt\left[\frac{1}{1-t}(a^{-t+1} - 1) - \frac{1}{t}(1 - a^{-t})\right], t > 0 \end{array} \tag{6}$$

Because
$$\lim_{t\to\infty}\left[\frac{1}{1-t}(a^{-t+1} - 1) - \frac{1}{t}(1 - a^{-t})\right] = 0,$$
from (6) it follows that
$$\lim_{t\to\infty} J(t) = \lim_{t\to\infty} U(t),$$
i.e. we have
$$\lim_{t\to\infty}\int_1^t \frac{f(x)}{x}dx = \lim_{t\to\infty}\int_1^a f(x^t)dt$$
and the desired result follows.

Remark. The relation (2) is a natural reformulation of the first part of Problem 5.183 in [3] proposed by the second author and S. Rădulescu.

3.Two applications

Application 1. Let us evaluate

$$\lim_{n\to\infty} n\int_1^a \frac{dx}{x^n+k}, \tag{7}$$

where $k>0$, $a>1$ are fixed real numbers.
Using the result in Theorem for function $f(x)=\frac{1}{x+k}$, $x\geq 1$, we obtain

$$\lim_{n\to\infty} n\int_1^a \frac{dx}{x^n+k}=\int_1^\infty \frac{dx}{x(x+k)}=\frac{1}{k}\ln\frac{x}{x+k}\Big|_1^\infty=\frac{1}{k}\ln(k+1).$$

Note that for $k=1$ we get

$$\lim_{n\to\infty} n\int_1^a \frac{dx}{x^n+1}=\ln 2,$$

i.e. the second part of Problem 5.183 in [3].

Application 2. Let us evaluate

$$\lim_{n\to\infty} n\int_0^1 \frac{x^{n-2}}{x^{2n}+x^n+1}dx, \tag{8}$$

which is a problem proposed by D. Popa to Mathematical Regional Contest "Grigore Moisil", 2002 (see [2] for details).
Fix $a\in(0,1)$ and we can write

$$n\int_0^1 \frac{x^{n-2}}{x^{2n}+x^n+1}dx=n\int_0^a \frac{x^{n-2}}{x^{2n}+x^n+1}dx+n\int_a^1 \frac{x^{n-2}}{x^{2n}+x^n+1}dx.$$

For the first term in the right side we have

$$0\leq n\int_0^a \frac{x^{n-2}}{x^{2n}+x^n+1}dx\leq n\int_0^a x^{n-2}dx=\frac{na^{n-1}}{n-1}\to 0.$$

For the second term we obtain

$$n\int_a^1 \frac{x^{n-2}}{x^{2n}+x^n+1}dx=n\int_a^1 \frac{x^n dx}{x^2(x^{2n}+x^n+1)}=n\int_1^{1/a} \frac{t^n}{t^{2n}+t^n+1}dt.$$

The function $f(t)=\frac{t}{t^2+t+1}$ satisfies $\lim\limits_{t\to\infty} tf(t)=1$ and we have

$$\int_1^\infty \frac{f(t)}{t}dt=\lim_{t\to\infty}\int_1^\infty \frac{dt}{t^2+t+1}=\frac{2}{\sqrt{3}}arctg\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\Big|_1^\infty=\frac{\pi}{3\sqrt{3}}$$

Applying the result in Theorem it follows that

$$\lim_{n\to\infty} n\int_0^1 \frac{x^{n-2}}{x^{2n}+x^n+1}dx=\frac{\pi}{3\sqrt{3}}.$$

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