METRIC SUBGROUPS OF ISOMETRIES ON AN ULTRAMETRIC SPACE

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Abstract. Let E be an ultrametric space, d the distance on E and G a group of bijective maps $\psi : E \to E$ which are isometries. We investigate properties of those subgroups H of G which are defined in terms of metric constraints, of the form

$$H = \{\psi \in G : d(x, \psi(x)) \leq f(x),\ x \in E\}$$

for some function $f : E \to [0, \infty]$.

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1.Introduction

Let E be an ultrametric space, that is a metric space on which the distance d satisfies the triangle inequality in the stronger form

$$d(x, z) \leq \max\{d(x, y),\ d(y, z)\}$$

for any $x, y, z \in E$. We denote by $\mathcal{F}_E$ the set of maps $f : E \to [0, \infty]$ and by $\mathcal{G}$ the group of bijective maps $\psi : E \to E$ which are isometries,

$$d(\psi(x), \psi(y)) = d(x, y)$$

for any $x, y \in E$. We say that a subgroup H of $\mathcal{G}$ is a metric subgroup provided that

$$H = \{\psi \in \mathcal{G} : d(x, \psi(x)) \leq f(x),\ x \in E\}$$

for some $f \in \mathcal{F}_E$. More generally, if G is a subgroup of $\mathcal{G}$, and H is a subgroup of G, we say that H is a metric subgroup of G if there exists a function $f \in \mathcal{F}_E$ such that

$$H = \{\psi \in G : d(x, \psi(x)) \leq f(x),\ x \in E\}.$$

In this definition G itself does not have to be a metric subgroup of $\mathcal{G}$, we only ask that H be defined inside G by metric constraints as above. So we may have situations when a subgroup H of $\mathcal{G}$ which is also a subgroup of G, fails to be a metric subgroup of $\mathcal{G}$ while at the same time H is a metric subgroup of G.

Various properties of groups of isometries on an ultrametric space have been investigated in [9], [10] and [11]. The starting point was the observation that, unlike in the case of a general metric space, if E is an ultrametric space then for any $f \in \mathcal{F}_E$ the set

$$\mathcal{G}(f) = \{\psi \in \mathcal{G} : d(x, \psi(x)) \leq f(x),\ x \in E\}$$

is a subgroup of $\mathcal{G}$. The motivation for considering these notions came from the theory of local fields (for a general presentation see [4] or [7]). If p is a prime number and $E = \mathbf{C_p}$ is the completion of the algebraic closure $\overline{\mathbf{Q}}_p$ of the field of $p-$adic numbers $\overline{\mathbf{Q}}_p$, then E is an ultrametric space and any automorphism $\sigma \in \mathrm{Gal}_{\mathrm{cont}}(E/\mathbf{Q}_p) \cong \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is an isometry. Some important groups of automorphisms, such as the ramification groups, can naturally be interpreted in the above metric framework. Also, by Galois theory in $\mathbf{C_p}$, as developed in [3], [6],[8], each closed subgroup H of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ corresponds to a closed subfield L of $\mathbf{C_p}$, given by

$$L = \{x \in \mathbf{C_p} : \sigma(\mathbf{x}) = \mathbf{x},\ \sigma \in \mathbf{H}\}.$$

We see that

$$H = \{\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) :\ d(x, \sigma(x)) \leq f(x),\ x \in E\}$$

with

$$f(x) = \begin{cases} 0 & \text{if } x \in L \\ \infty & \text{if } x \in E \setminus L\,. \end{cases}$$

Thus, with the above definition, H is a metric subgroup of $G = \mathrm{Gal}_{\mathrm{cont}}(\mathbf{C_p}/\mathbf{Q_p})$. Clearly one can define the same subgroup H with the aid of other functions from $\mathcal{F}_E$. If we choose for instance a generating element

T of L (see [5],[1],[2]), that is an element $T \in L$ for which $\mathbf{Q_p[T]}$ is dense in L, then we may replace the above f by

$$g(x) = \begin{cases} 0 & \text{if } x = T \\ \infty & \text{if } x \neq T \end{cases}$$

without changing the group H. In the present paper we work with a general ultrametric space E, a subgroup G of $\mathcal{G}$, and we discuss some properties of metric subgroups H of G.

2.Groups of isometries and m.l.c. functions

The notion of metric locally constant function (m.l.c. for short) was introduced in [9], in order to investigate certain groups of isometries on a given ultrametric space, and was subsequently studied also in [10] and [11]. In this section we collect some results from [9] concerned with metric locally constant functions and the corresponding groups of isometries.

Let E be an ultrametric space, d the distance on E and $\mathcal{F}_E = \{f : E \to [0, \infty]\}$. For any $x \in E$ and $r > 0$ we denote by $B(x, r)$ the open ball of radius r centered at x. A function $f \in \mathcal{F}_E$ is said to be *metric locally constant* provided that for any $x \in E$ and any $y \in B(x, f(x))$ one has $f(y) = f(x)$. We denote by $\tilde{\mathcal{F}}_E$ the set of *m.l.c.* functions.

Let $f \in \tilde{\mathcal{F}}_E$. Then:

(i) f is constant equal to ∞ or $Im\, f \subseteq [0, \infty)$.

(ii) f is locally constant on the open set $E \setminus f^{-1}(0)$.

(iii) If $f(x_0) = 0$ then $f(x) \leq d(x, x_0)$ for any $x \in E$.

(iv) f is continuous.

For any $z \in E$ denote by d_z the function given by $d_z(x) = d(x, z)$ for any $x \in E$.

Theorem 1. *$\tilde{\mathcal{F}}_E$ contains the constant functions, the d_z's and it is closed under taking* inf, sup *and under scalar multiplication with numbers $c \in [0, 1]$.*

Corollary 1. *For any subset A of E, the function $d_A : E \to [0, \infty)$ given by $d_A(x) = d(x, A) = \inf_{y \in A} d(x, y)$ for any $x \in E$, is m.l.c.*

One has the following structure theorem for $\tilde{\mathcal{F}}_E$.

Theorem 2. *$\tilde{\mathcal{F}}_E$ coincides with the smallest subset of $\mathbf{F_E}$ which contains the constants, the d_z's and is closed under taking inf and sup.*

Let us define for any $f \in \mathcal{F}_E$ a new element $\tilde{f} \in \mathcal{F}_E$ given by

$$\tilde{f}(x) = \inf_{y\in E}\max\{d(x,y), f(y)\}$$

for any $x \in E$. If we denote by c_t the constant function $c_t(x) = t$, then the above equality can also be written in the form

$$\tilde{f} = \inf_{y\in E}\max\{d_y, c_{f(y)}\}.$$

Some properties of the map which associates $\tilde{f}$ to f are collected in the following result.

THEOREM 3. *The map from $\mathcal{F}_E$ to $\mathcal{F}_E$ given by $f \mapsto \tilde{f}$ has the following properties:*
(i) *If $f \leq g$ then $\tilde{f} \leq \tilde{g}$.*
(ii) If $\tilde{f} = g$ then $\tilde{g} = g$.
(iii) *$\tilde{f} \leq f$ for any $f \in \mathcal{F}_E$.*
(iv) *$\tilde{\mathcal{F}}_E = \{\tilde{f} \in \mathcal{F}_E : f \in \mathcal{F}_E\} = \{f \in \mathcal{F}_E : f = \tilde{f}\}$.*
(v) *If H is a subset of $\mathcal{F}_E$ and $f(x) = \inf_{h\in H}h(x)$ for any $x \in E$, then $\tilde{f}(x) = \inf_{h\in H}\tilde{h}(x)$ for any $x \in E$.*

Let $\mathcal{G}$ be the group of bijective maps $\psi : E \to E$ which are isometries. For any $f \in \mathcal{F}_E$ consider the set

$$\mathcal{G}(f) = \{\psi \in \mathcal{G} : d(x, \psi(x)) \leq f(x), x \in E\}.$$

For a general metric space E, $\mathcal{G}(f)$ might or might not be a subgroup of $\mathcal{G}$. For an ultrametric space E, $\mathcal{G}(f)$ is a subgroup of $\mathcal{G}$ for any $f \in \mathcal{F}_E$.

THEOREM 4. (i)*For any $f \in \mathcal{F}_E$, $\mathcal{G}(f)$ is a subgroup of $\mathcal{G}$.*
(ii)*If $f \leq g$ then $\mathcal{G}(f)$ is a subgroup of $\mathcal{G}(g)$.*
(iii)*For any subset H of $\mathcal{F}_E$ one has $\mathcal{G}(\inf_{h\in H}h) = \bigcap_{h\in H}\mathcal{G}(h)$.*
(iv)*$\mathcal{G}(\tilde{f}) = \mathcal{G}(f)$ for any $f \in \mathcal{F}_E$.*
(v)*If $f, g \in \mathcal{F}_E$ are such that $\tilde{f} \leq g$ and $\tilde{g} \leq f$, then $\mathcal{G}(f) = \mathcal{G}(g)$.*

Property (iv), together with the above formulas for $\tilde{f}$, give an explicit way of deforming a given $f \in \mathcal{F}_E$ (to make it metric locally constant), without changing the group $\mathcal{G}(f)$. Property (v) produces instances when one can conclude that two functions f, g, which might be related in a complicated metric way, produce the same group of isometries.

3. Metric subgroups of isometries

Notations being as in the previous sections, let now G be any subgroup of $\mathcal{G}$, and let H be a subgroup of G. We say that H is a metric subgroup of G provided that there exists a function $f \in \mathcal{F}_E$ such that

$$H = \{\psi \in G : d(x, \psi(x)) \leq f(x), x \in E\}.$$

We note that not all the groups of isometries on ultrametric spaces are metric subgroups.

Indeed, let us consider the following example. Let E consist of n elements and let the distance d be given by $d(x, y) = 1$ for any two distinct elements x, y of E. Then any bijective map $\psi : E \to E$ is an isometry, so $\mathcal{G}$ consists of all the permutations of the set E. Fix $f \in \mathcal{F}_E$ and let $\psi \in \mathcal{G}(f)$. For any $x \in E$ for which $f(x) < 1$, the inequality $d(x, \psi(x)) \leq f(x)$ forces the equality $\psi(x) = x$. For any $x \in E$ with $f(x) \geq 1$, the inequality $d(x, \psi(x)) \leq f(x)$ is automatically satisfied. It follows that $\psi \in \mathcal{G}(f)$ if and only if ψ invariates every element of $f^{-1}([0, 1))$. Therefore in this example the metric subgroups of $\mathcal{G}$ are in one-to-one correspondence with the subsets of E. Namely, for any subset S of E we have a metric subgroup of $\mathcal{G}$, consisting of all the permutations of E which invariate every element of S. Thus in particular the subgroup generated by a cyclic permutation of length $m \geq 3$, $m \leq n$, will not be a metric subgroup of $\mathcal{G}$.

Returning to the general case, we gather some properties of metric subgroups in the following theorem.

Theorem 5. *Let E be an ultrametric space and denote by $\mathcal{G}$ the group of bijective isometries on E.*

(i) *If G is a subgroup of $\mathcal{G}$ and $(H_j)_{j \in J}$ is a family of metric subgroups of G, then their intersection $\cap_{j \in J} H_j$* is a metric subgroup of G.

(ii) *If $H \subseteq F \subseteq G$ are subgroups of $\mathcal{G}$ and H is a metric subgroup of G, then H is a metric subgroup of F.*

(iii) *If $H \subseteq F \subseteq G$ are subgroups of $\mathcal{G}$, H is a metric subgroup of F and F is a metric subgroup of G, then H is a metric subgroup of G.*

(iv) *If G is a subgroup of $\mathcal{G}$ and H is a metric subgroup of G, then for any subgroup F of $\mathcal{G}$, $H \cap F$ is a metric subgroup of $G \cap F$.*

(v) *For any subgroup G of $\mathcal{G}$, and any subgroup H of G, there is a smallest metric subgroup of G which contains H. We denote this subgroup by H_G.*

(vi) *For any subgroup G of $\mathcal{G}$, and any subgroups H, F of G, one has*

$$(H \cap F)_G \subseteq H_G \cap F_G.$$

(vii)*For any subgroup H of $\mathcal{G}$, and any subgroups F, G of $\mathcal{G}$ which contain H, one has*

$$H_{(G \cap F)} = H_G \cap H_F.$$

Proof. (i). Let G be a subgroup of $\mathcal{G}$. Let $(H_j)_{j \in J}$ be a family of metric subgroups of G and denote their intersection by H. Next, for any $j \in J$ choose a function $f_j \in \mathcal{F}_E$ such that

$$H_j = \{\psi \in G : d(x, \psi(x)) \leq f_j(x), x \in E\}.$$

Define $f \in \mathcal{F}_E$ by

$$f(x) = \inf\{f_j(x) : j \in J\}$$

for any $x \in E$. Then

$$H = \{\psi \in G : d(x, \psi(x)) \leq f_j(x), x \in E, j \in J\}$$

$$= \{\psi \in G : d(x, \psi(x)) \leq f(x), x \in E\},$$

so H is a metric subgroup of G.

(ii). Let $H \subseteq F \subseteq G$ be subgroups of $\mathcal{G}$ such that H is a metric subgroup of G. Choose a function $f \in \mathcal{F}_E$ for which

$$H = \{\psi \in G : d(x, \psi(x)) \leq f(x), x \in E\}.$$

Then for any $\psi \in H$ and any $x \in E$ we have $d(x, \psi(x)) \leq f(x)$, while for any ψ which belongs to G but not to H, in particular for any ψ which belongs to F but not to H, there exists an element $y \in E$, depending on ψ, for which $d(y, \psi(y)) > f(y)$. It follows that

$$H = \{\psi \in F : d(x, \psi(x)) \leq f(x), x \in E\},$$

hence H is a metric subgroup of F.

(iii). Let $H \subseteq F \subseteq G$ be subgroups of $\mathcal{G}$, such that H is a metric subgroup of F, and F is a metric subgroup of G. Choose functions $f, g \in \mathcal{F}_E$ such that

$$H = \{\psi \in F : d(x, \psi(x)) \le f(x), x \in E\},$$

and

$$F = \{\psi \in G : d(x, \psi(x)) \le g(x), x \in E\}.$$

Consider the function $h \in \mathcal{F}_E$ given by

$$h(x) = \min\{f(x), g(x)\}$$

for all $x \in E$. Then, on the one hand, for any $\psi \in H$ and any $x \in E$ we have $d(x, \psi(x)) \le h(x)$ since ψ belongs to both H and F, and on the other hand, if $\psi \in G$ is such that $d(x, \psi(x)) \le h(x)$ for any $x \in E$, then $\psi \in F$, and further $\psi \in H$. Therefore

$$H = \{\psi \in G : d(x, \psi(x)) \le h(x), x \in E\},$$

so H is a metric subgroup of G.

(iv). Let G, F be subgroups of $\mathcal{G}$, and let H be a metric subgroup of G. Choose a function $f \in \mathcal{F}_E$ such that

$$H = \{\psi \in G : d(x, \psi(x)) \le f(x), x \in E\}.$$

If ψ is an element of $H \cap F$, then clearly $d(x, \psi(x)) \le f(x)$ for all $x \in E$. If ψ is an element of $G \cap F$, and ψ satisfies the inequalities $d(x, \psi(x)) \le f(x)$ for all $x \in E$, then, since ψ belongs to G, it follows that ψ belongs to H, so ψ belongs to $H \cap F$. In conclusion

$$H \cap F = \{\psi \in G \cap F : d(x, \psi(x)) \le f(x), x \in E\},$$

which shows that $H \cap F$ is a metric subgroup of $G \cap F$.

(v). Let G be a subgroup of $\mathcal{G}$, and let H be a subgroup of G. Consider the set $\mathcal{H}$ of all the metric subgroups of G which contain H. Note that G is a metric subgroup of itself, since for instance for the constant function $f(x) = \infty$ for all $x \in E$ we have

$$G = \{\psi \in G : d(x, \psi(x)) \le f(x), x \in E\}.$$

Thus G belongs to $\mathcal{H}$, so $\mathcal{H}$ is nonempty. By property (i) above we know that the intersection of all the subgroups of G which belong to $\mathcal{H}$ is also a metric subgroup of G, so it belongs to $\mathcal{H}$. Evidently this subgroup is the smallest metric subgroup of G which contains H.

(vi). Let G be a subgroup of $\mathcal{G}$, and let H, F be subgroups of G. We know that H_G is a metric subgroup of G which contains H, so it contains $H \cap F$. We also know that $(H \cap F)_G$ is the smallest metric subgroup of G which contains $H \cap F$. Hence $(H \cap F)_G$ is a subgroup of H_G. By a similar reasoning it follows that $(H \cap F)_G$ is a subgroup of F_G. Hence

$$(H \cap F)_G \subseteq H_G \cap F_G,$$

which proves (vi).

(vii). Let H be a subgroup of $\mathcal{G}$, and let F, G be subgroups of $\mathcal{G}$ which contain H. By definition we know that H_G is a metric subgroup of G which contains H. Taking intersections with F, and using property (iv) above, we derive that $H_G \cap F$ is a metric subgroup of $G \cap F$ which contains H. On the other hand, $H_{(G\cap F)}$ is the smallest metric subgroup of $G \cap F$ which contains H. It follows that $H_{(G\cap F)}$ is contained in $H_G \cap F$, so it is contained in H_G. Similarly we find that $H_{(G\cap F)}$ is contained in H_F. Thus

$$H_{(G\cap F)} \subseteq H_G \cap H_F.$$

Suppose this inclusion is strict. Then there exists an isometry σ such that σ belongs to both H_G and H_F, and σ does not belong to $H_{(G\cap F)}$. We claim that

$$d(x, \sigma(x)) \leq \sup\{d(x, \psi(x)) : \psi \in H\},$$

for any $x \in E$. Indeed, if this inequality fails for some element $y \in E$, then let us consider the group

$$G(f) = \{\psi \in G : d(x, \psi(x)) \leq f(x), x \in E\},$$

where $f \in \mathcal{F}_E$ is defined by

$$f(x) = \sup\{d(x, \psi(x)) : \psi \in H\},$$

for all $x \in E$. Note that σ does not belong to $G(f)$ since $d(y, \psi(y)) > f(y)$. On the other hand $G(f)$ is a metric subgroup of G which contains H. Therefore H_G is contained in $G(f)$, and since σ does not belong to $G(f)$, it would follow that σ does not belong to H_G, contrary to our assumptions. This proves the claim.

Let now $h \in \mathcal{F}_E$ be a function for which

$$H_{(G\cap F)} = \{\psi \in G \cap F : d(x, \psi(x)) \le h(x), x \in E\}.$$

Then we have

$$f(x) = \sup\{d(x, \psi(x)) : \psi \in H\} \le h(x)$$

for any $x \in E$, and from the above claim it follows that $d(x, \sigma(x)) \le f(x) \le h(x)$ for all $x \in E$. This in turn implies that $\sigma \in H_{(G\cap F)}$, contrary to our assumptions on σ. In conclusion we have the equality

$$H_{(G\cap F)} = H_G \cap H_F,$$

and this completes the proof of the theorem.

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