SOME SUBCLASSES OF n -UNIFORMLY CLOSE TO CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract.In this paper we define some subclasses on n-uniformly close to convex functions with negative coefficients and we obtain necessary and sufficient conditions and some other properties regarding this classes.

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1.Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U = \{z \in \mathbf{C} : |z| < 1\}$, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$ and $S = \{f \in A : f \text{ is univalent in } U\}$.

We recall here the definition of the well - known class of starlike functions:

$$S^* = \left\{ f \in A : Re\frac{zf'(z)}{f(z)} > 0 \ , \ z \in U \right\}.$$

In [8] the subfamily T of S consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \ a_j \geq 0, j = 2, 3, ..., \ z \in U. \quad (1)$$

was introduced.

Theorem 1.[7] *If* $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...$, $z \in U$ *then the next assertions are equivalent*:

(i) $\sum\limits_{j=2}^{\infty} j a_j \leq 1$

(ii) $f \in T$

(iii) $f \in T^*$, *where* $T^* = T \bigcap S^*$ *and* S^* *is the well-known class of starlike functions.*

Let D^n be the Salagean differential operator (see [6]) $D^n : A \to A$, $n \in \mathbf{N}$, defined as:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z)).$$

Remark 1.*If* $f \in T$, $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $z \in U$ *then* $D^n f(z) = z - \sum\limits_{j=2}^{\infty} j^n a_j z^j$.

Let consider the generalized Alexander operator $I^\lambda : A \to A$ defined as:

$$I^\lambda f(z) = z + \sum_{j=2}^{\infty} \frac{1}{j^\lambda} a_j z^j \ , \lambda \in \mathbf{R}\, , \lambda \geq 0\, , \tag{2}$$

where $f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$.

For $\lambda = 1$ we obtain the Alexander integral operator.

The purpose of this note is to define, using the Salagean differential operator, some subclasses on n-uniformly close to convex functions with negative coefficients and necessary and sufficient conditions and some preserving properties of the generalized Alexander operator regarding this classes.

2.Preliminary results

Let $k \in [0, \infty)$, $n \in \mathbf{N}^*$. We define the class $(k, n) - S^*$ (see the definition of the class $(k, n) - ST$ in [2]) by $f \in S^*$ and

$$\mathrm{Re} \frac{D^n f(z)}{f(z)} > k \frac{D^n f(z)}{f(z)} - 1\, , \, z \in U\, .$$

Remark 2.*(for more details see [2]).We denote by* p_k , $k \in [0,\infty)$ *the function which maps the unit disk conformally onto the region* Ω_k*, such that* $1 \in \Omega_k$ *and*

$$\partial\Omega_k = u + iv \,:\, u^2 = k^2(u-1)^2 + k^2v^2 \,.$$

The domain Ω_k *is elliptic for* $k > 1$*, hyperbolic when* $0 < k < 1$*, parabolic for* $k = 1$*, and a right half-plane when* $k = 0$*. In this conditions, a function* f *is in the class* $(k,n) - S^*$ *if and only if* $\frac{D^n f(z)}{f(z)} \prec p_k$ *or* $\frac{D^n f(z)}{f(z)}$ *take all values in the domain* Ω_k *.*

In [1] is defined the class $(k,n) - CC$ thus:

Definition 1. *Let* $f \in A$*,* $k \in [0,\infty)$ *and* $n \in \mathbf{N}^*$*. We say that the function* f *is in the class* $(k,n)-CC$ *with respect to the function* $g \in (k,n)-S^*$ *if*

$$\mathrm{Re}\frac{D^n f(z)}{g(z)} > k \cdot \left|\frac{D^n f(z)}{g(z)} - 1\right| , \; z \in U \,.$$

Remark 3.*Geometric interpretation:* $f \in (k,n) - CC$ *with respect to the function* $g \in (k,n) - S^*$ *if and only if* $\frac{D^n f(z)}{g(z)} \prec p_k$ *(see Remark 1.) or* $\frac{D^n f(z)}{g(z)}$ *take all values in the domain* Ω_k *(see Remark 1).*

Remark 4.*From the geometric properties of the domains* Ω_k *we have that* $(k_1,n) - CC \subset (k_2,n) - CC$*, where* $k_1 \geq k_2$*.*

3.Main results

Definition 2.*We define the class* $(k,n) - T^*$*, where* $k \in [0,\infty)$ *and* $n \in \mathbf{N}^*$*, through*

$$(k,n) - T^* \;=\; (k,n) - S^* \bigcap T \,.$$

Theorem 2.*Let* f *of the form (1),* $k \in [0,\infty)$ *and* $n \in \mathbf{N}^*$*. Then* $f \in (k,n) - T^*$ *if and only if*

$$\sum_{j=2}^{\infty} [j^n(k+1) - k]a_j < 1. \tag{3}$$

Proof. Let $f \in (k,n) - T^*$ with $k \in [0,\infty)$ and $n \in \mathbf{N}^*$. We have

$$\mathrm{Re}\frac{D^n f(z)}{f(z)} > k \cdot \left|\frac{D^n f(z)}{f(z)} - 1\right| , \; z \in U \,.$$

If we take $z \in [0,1)$, we have (see Remark 1)

$$\frac{1-\sum\limits_{j=2}^{\infty} j^n a_j z^{j-1}}{1-\sum\limits_{j=2}^{\infty} a_j z^{j-1}} > k \cdot \frac{\sum\limits_{j=2}^{\infty} (j^n-1) a_j z^{j-1}}{1-\sum\limits_{j=2}^{\infty} a_j z^{j-1}} \tag{4}$$

From $f \in (k,n) - T^*$ we have (see Theorem 1)

$$\sum_{j=2}^{\infty} a_j z^{j-1} \le \sum_{j=2}^{\infty} j a_j z^{j-1} < \sum_{j=2}^{\infty} j a_j \le 1$$

and thus

$$\sum_{j=2}^{\infty} a_j z^{j-1} < 1$$

or

$$\left|1-\sum_{j=2}^{\infty} a_j z^{j-1}\right| = 1-\sum_{j=2}^{\infty} a_j z^{j-1}\,.$$

In this conditions from (4) we obtain

$$1-\sum_{j=2}^{\infty} j^n a_j z^{j-1} > k \cdot \sum_{j=2}^{\infty} (j^n-1) a_j z^{j-1}\,.$$

Letting $z \to 1^-$ along the real axis we have

$$\sum_{j=2}^{\infty} j^n(k+1) - k a_j < 1\ .$$

Now let $f \in T$ for which the relation (3) hold.
The relation $\mathrm{Re}\frac{D^n f(z)}{f(z)} > k\left|\frac{D^n f(z)}{f(z)} - 1\right|$ is equivalent with

$$k\left|\frac{D^n f(z)}{f(z)} - 1\right| - \mathrm{Re}\left(\frac{D^n f(z)}{f(z)} - 1\right) < 1. \tag{5}$$

Using Remark 1 and Theorem 1 we have

$$k\left|\frac{D^n f(z)}{f(z)} - 1\right| - \mathrm{Re}\left(\frac{D^n f(z)}{f(z)} - 1\right) \le (k+1)\left|\frac{D^n f(z)}{f(z)} - 1\right| \le$$

$$\leq (k+1)\frac{\sum\limits_{j=2}^{\infty}(j^n-1)a_j\,|z|^{j-1}}{1-\sum\limits_{j=2}^{\infty}a_j\,|z|^{j-1}} \leq (k+1)\frac{\sum\limits_{j=2}^{\infty}(j^n-1)a_j}{1-\sum\limits_{j=2}^{\infty}a_j}.$$

Using (3) we have $(k+1)\frac{\sum\limits_{j=2}^{\infty}(j^n-1)a_j}{1-\sum\limits_{j=2}^{\infty}a_j} < 1$ and thus the condition (5) hold.

Definition 3. *Let $f \in T$, $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...$, $z \in U$. We say that f is in the class $(k,n) - CCT$, $k \in [0,\infty)$, $n \in \mathbf{N}^*$, with respect to the function $g \in (k,n) - T^*$, if*

$$\mathrm{Re}\frac{D^n f(z)}{g(z)} > k \cdot \frac{D^n f(z)}{g(z)} - 1, z \in U.$$

Theorem 3. *Let $k \in [0,\infty)$ and $n \in \mathbf{N}^*$. The function f of the form (1) is in $(k,n)-CCT$, with respect to the function $g \in (k,n)-T^*$, $g(z) = z-\sum\limits_{j=2}^{\infty} b_j z^j$, $b_j \geq 0$, $j = 2, 3, ...$, if and only if*

$$\sum_{j=2}^{\infty}\left[(k+1)\cdot|j^n a_j - b_j| + b_j\right] < 1. \tag{6}$$

Proof. Let $f \in (k,n) - CCT$, where $k \in [0,\infty)$ and $n \in \mathbf{N}^*$, with respect to the function $g \in (k,n) - T^*$. We have

$$\mathrm{Re}\frac{D^n f(z)}{g(z)} > k \cdot \frac{D^n f(z)}{g(z)} - 1, z \in U.$$

If we take $z \in [0,1)$, we have (see Remark 1)

$$\frac{1-\sum\limits_{j=2}^{\infty} j^n a_j z^{j-1}}{1-\sum\limits_{j=2}^{\infty} b_j z^{j-1}} > k \cdot \frac{\left|\sum\limits_{j=2}^{\infty}\left[j^n a_j - b_j\right] z^{j-1}\right|}{\left|1-\sum_{j=2}^{\infty} b_j z^{j-1}\right|} \tag{7}$$

From $g \in (k,n) - T^*$ we have (see Theorem 1)

$$1-\sum_{j=2}^{\infty} b_j z^{j-1} > 0$$

and thus from (7) we obtain

$$1-\sum_{j=2}^{\infty} j^n a_j z^{j-1} > k\cdot \sum_{j=2}^{\infty} j^n a_j - b_j z^{j-1}\,.$$

Letting $z\to 1^-$ along the real axis we have

$$1-\sum_{j=2}^{\infty} j^n a_j > k\cdot\left|\sum_{j=2}^{\infty}(j^n a_j - b_j)\right|$$

and thus

$$k\cdot\sum_{j=2}^{\infty} j^n a_j - b_j + \sum_{j=2}^{\infty} j^n a_j - 1 < 0 \tag{8}$$

From $k\cdot\left|\sum\limits_{j=2}^{\infty} j^n a_j - b_j\right| + \sum\limits_{j=2}^{\infty} j^n a_j - 1 \le$

$$\le k\cdot\sum_{j=2}^{\infty}|j^n a_j - b_j| + \sum_{j=2}^{\infty} j^n a_j - 1 = \sum_{j=2}^{\infty}\left[k\cdot|j^n a_j - b_j| + j^n a_j\right] - 1$$

we obtain the condition

$$\sum_{j=2}^{\infty}\left[k\cdot|j^n a_j - b_j| + j^n a_j\right] < 1 \tag{9}$$

which implies (8). It is easy to observe that if (6) hold then the inequality (9) is true.

Now let take $g\in(k,n)-T^*$, where $k\in[0,\infty)$, $n\in\mathbf{N}^*$, and $f\in T$ for which the relation (6) hold.

The relation

$$\mathrm{Re}\left(\frac{D^n f(z)}{g(z)}\right) > k\left|\frac{D^n f(z)}{g(z)} - 1\right|$$

is equivalent with

$$k\left|\frac{D^n f(z)}{g(z)} - 1\right| - \mathrm{Re}\left(\frac{D^n f(z)}{g(z)} - 1 < 1\right). \tag{10}$$

Using Remark 1 and Theorem 1 we have

$$k\left|\frac{D^n f(z)}{g(z)}-1\right|-\operatorname{Re}\left(\frac{D^n f(z)}{g(z)}-1\right)\leq (k+1)\left|\frac{D^n f(z)}{g(z)}-1\right|\leq$$

$$\leq (k+1)\frac{\sum\limits_{j=2}^{\infty}\left|j^n a_j-b_j\right|\cdot |z|^{j-1}}{1-\sum\limits_{j=2}^{\infty} b_j\,|z|^{j-1}}\leq (k+1)\frac{\sum\limits_{j=2}^{\infty}\left|j^n a_j-b_j\right|}{1-\sum\limits_{j=2}^{\infty} b_j}\,.$$

Using (6) we have

$$(k+1)\frac{\sum\limits_{j=2}^{\infty}\left|j^n a_j-b_j\right|}{1-\sum\limits_{j=2}^{\infty} b_j}<1$$

and thus the condition (10) hold.

THEOREM 4.*Let $k\in[0,\infty)$ and $n\in \mathrm{N}^*$. If $F(z)\in(k,n)-T^*$ and I^λ is the generalized Alexander operator defined by (2) then $f(z)=I^\lambda(F)(z)\in (k,n)-T^*$.*

Proof. From $F(z)\in(k,n)-T^*$, $F(z)=z-\sum\limits_{j=2}^{\infty}a_j z^j$, $a_j\geq 0$, $j=2,3,...$, we have (see Theorem 2)

$$\sum_{j=2}^{\infty}\left[j^n(k+1)-k\right]a_j<1. \tag{11}$$

From (2) we have $f(z)=I^\lambda(F)(z)=z-\sum\limits_{j=2}^{\infty}{}_j z^j$, where $\alpha_j=\frac{1}{j^\lambda}a_j\geq 0$, $j\geq 2$, $\lambda\in \mathrm{R}$, $\lambda\geq 0$.

By using Theorem 2 it is sufficient to prove that

$$\sum_{j=2}^{\infty} j^n(k+1)-k_j<1. \tag{12}$$

We have

$$\sum_{j=2}^{\infty}\left[j^n(k+1)-k\right]\alpha_j=\sum_{j=2}^{\infty}\left[j^n(k+1)-k\right]\left(\frac{1}{j}\right)^{\lambda}a_j\sum_{j=2}^{\infty}\left[j^n(k+1)-k\right]a_j, \tag{13}$$

where $a_j \geq 0$, $j \geq 2$,$k \in [0,\infty)$ and $n \in \mathrm{N}^*$.

From (13) and (11) we obtain the condition (12)and thus $f(z) \in (k,n)-T^*$.

In a similarly way we prove the next theorem:

Theorem 4. *Let $k \in [0,\infty)$, $n \in \mathbf{N}^*$, $\lambda \in \mathbf{R}$ and $\lambda \geq 0$. If $F(z) \in (k,n)-CCT$ with respect to the function $G(z) \in (k,n)-T^*$ and I^λ is the generalized Alexander operator defined by (2) then $f(z) = I^\lambda(F)(z) \in (k,n)-CCT$ with respect to the function $g(z) = I^\lambda(G)(z) \in (k,n)-T^*$ (see the above theorem).*

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