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ON A THEOREM WHICH LIMITS THE NUMBER OF MIXED INTERPOLATED DERIVATIVES FOR SOME PLANE REGULAR UNIFORM RECTANGULAR BIRKHOFF INTERPOLATION SCHEMES

by
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Abstract: As the title suggest, on this paper we will establish a theorem which limits the number of choices of interpolated mixed derivatives (so also of interpolation derivatives) in order to obtain uniform rectangular Birkhoff interpolation schemes, and which makes possible their regularity.
In the first part of the paper ((1)-(7)) we present specific notions that can be studied in more detail by consulting the bibliography. These concepts are mentioned here because they are the basic suport for the theorem. The final remarks refer to certain limit situations which follow from this theorem.

In this paper we will use the following concepts and notations:

(1) A finite set $L \subset IN^2$ is *inferior* if $R(u,v) \subset L$ for any $(u,v) \in L$, where

$$R(u,v) = \{(i,j) \in IN^2 : i \le u,\ j \le v\}.$$

(2) A set of nodes Z is (p,q)*-rectangular* ($p,q \in IN$) if it can be written as

$$Z = \{(x_i, y_j) : 0 \le i \le p, 0 \le j \le q)\},$$

where $x_0, x_1, ...x_p$ are real pair-wise distinctive numbers, and the same applies to $y_0, y_1, ...y_q$. If such numbers as p and q exist, we say shortly that Z is *rectangular*.

(3) A *bidimensional uniform Birkhoff* interpolation scheme is the triplet (Z,S,A) consisting of a set (of nodes)

$$Z = \{\mathbf{z}_t = (x_t, y_t) \in IR^2\}_{t=1}^{n},$$

an inferior set $S \subset IN^2$ and a subset A (of derivatives) of S. The Birkhoff interpolation problem associated to this scheme consists in determining the polynomials

$$P \in P_S = \left\{ P \in IR[x,y] : P(\mathbf{z}) = \sum_{(i,j)\in S} a_{ij} x^i y^j, \mathbf{z} = (x,y) \in IR^2 \right\},$$

which satisfy the equations:

$$\frac{\partial^{\alpha+\beta} P}{\partial x^{\alpha} \partial y^{\beta}}(\mathbf{z}_t) = c_{\alpha,\beta}(\mathbf{z}_t), (\forall)(\alpha,\beta) \in A, \mathbf{z}_t \in Z,$$

where $c_{\alpha,\beta}(\mathbf{z}_t)$ are arbitrary real constants. If Z has a rectangular shape, then (Z,S,A) is *U R (uniform rectangular) Birkhoff*.

(4) An interpolation scheme (Z,S,A) is called *normal* if

$$|Z||A| = |S|,$$

(where $|Z|$, $|A|$ and $|S|$ represent the cardinality of the corresponding sets). In case of normality, we denote by $D(Z,S,A)$ the determinant of the interpolation scheme.

(5) A (Z,S,A) scheme is *regular* (or (S,A) is *regular with respect to* Z), if $D(Z,S,A) \neq 0$. The scheme (S,A) is *regular* if (Z,S,A) is *regular* for any Z. If there exists at least one Z such that (Z,S,A) is *regular,* then (S,A) is *almost regular.* Next we denote an (almost) regular UR Birkhoff scheme by *RUR Birkhoff.*

(6) A stronger *Pólya*-type condition than the usual Pólya condition (see [1] or [2]), which is necessary for the almost regularity of a UR Birkhoff scheme, is: if (S,A) is almost regular, being regular with respect to a set of nodes (p,q)-rectangular, then for any inferior set $L \subset S$ one has

$$(p+1)(q+1)|A \cap L|$$
$$\geq |L| + pq|A \cap \partial L| + (p+q)|A \cap \partial_e L| + p|A \cap \partial_y L| + q|A \cap \partial_x L|$$

(see [1], theorem 4.6, page 11).

(7) The *necessary* conditions (criteria) for the almost regularity of an interpolation scheme (S,A) are the normality condition and the Pólya condition.

A_x and A_y denote the set of elements of A lying on the Ox axis, respectively Oy. Also $S_x = S \cap Ox$, and $S_y = S \cap Oy$, and one denotes by *mix(A)* the number of mixed derivatives defined by A (i.e. the number of elements $(\alpha,\beta) \in A$ with $\alpha \neq 0$ and $\beta \neq 0$).

Theorem: *If* (Z, S, A) *is a RUR Birkhoff interpolation scheme, then*

$$\sqrt{mix(A)} \leq \sqrt{|A|} - 1.$$

Proof: We use the inequality:

$$|A| \leq |A_x||A_y|,$$

proved in [1] (theorem 7.1, page 17), which follows from $|S| \leq |S_x||S_y|$ (S is inferior), $|S| = (p+1)(q+1)|A|$ (according to (2), the scheme considered in theorem is (p,q)-rectangular and is normal according to (4)) and from (6) for $L = S_x$, respectively $L = S_y$.

To get simpler formulas, we write $a = |A_x|$, $b = |A_y|$, and m - the number of mixed derivatives defined by A. With these notations, it is obvious that

$$|A| = m + a + b - 1.$$

Using this relation, the average inequality applied to the $a+b$ term and the previous-mentioned inequality ($|A| \leq ab$), we deduce that

$$|A| \geq m + 2\sqrt{ab} - 1 \geq m + 2\sqrt{|A|} - 1,$$

i.e.

$$(\sqrt{|A|} - 1)^2 \geq m,$$

which is equalivalent to the inequality from the statement of the theorem □

Consequences: 1. In any RUR Birkhoff interpolation scheme (Z, S, A) that interpolates mixed derivatives, one has $|A| \geq 4$.

2. If $|A| = |A_x||A_y|$ then $|S_x| = (p+1)|A_x|$, $|S_q| = (q+1)|A_y|$, and

$$S = R(p', q'),$$

where $p' = (p+1)|A_x| - 1$, $q' = (q+1)|A_y| - 1$.

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