FIXED POINTS AND MINIMAX INEQUALITIES

by
Mircea Balaj and Daniel Erzse

Abstract. Using Fan-Glicksberg fixed point theorem we obtain in this paper a fixed point theorem for the composition of two Kakutani maps.As application of this we get a new fixed point theorem, section properties and minimax inequalities.

1. Introduction

In order to give a simple proof for von Neuman minimax theorem, Kakutani [11] extended the well-known Brower's fixed point theorem to the case of an upper semicontinuous map T of a n-disk into itself. In turn, Kakutani's theorem was extended to Banach spaces by Bohnenblust and Karlin [4] and to locally convex Hausdorf topological vector spaces by Fan [5] and Glicksberg [8].

Using Fan-Glicksberg fixed point theorem we obtain in this paper a fixed point theorem for the composition of two Kakutani maps. As application of this we get a new fixed point theorem, section properties and minimax inequalities. Our results seem to be new altough they are closely related to some known results

2. Preliminaries

A *map* (or a *multifunction*) $T : X \to Y$ is a function from a set X into the power set 2^Y of Y; that is, a function with the values $T(x) \subset Y$ for $x \in X$ and the fibers $T^-(y) = \{x \in X : y \in T(x)\}$for $y \in Y$. Given two maps $S : X \to Y, T : Y \to Z$ then composition $T \circ S : X \to Z$ is defined by
$(T \circ S)(x) = T(S(x)) = \cup\{T(y) : y \in S(x)\}$.

Let X and Y be topological spaces. A map $T : X \to Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $F \subset Y$ the lower inverse of F under T, that is $T^-(F) = \{x \in X : T(x) \cap F \neq \phi\}$ is a closed subset of X or, equivalently, if for each open set $G \subset Y$, the upper inverse of G under T, that is $T^+(G) = \{x \in X : T(x) \subset F\}$ is an open subset of X. Note that if Y is compact Hausdorff and $T(x)$ is closed for each $x \in X$, then T is upper semicontinuous if and only if the graph of T, that is $\{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$. Recall also that the composition and the product of two u.s.c. are u.s.c., too.

If X is a topological space and Y is a convex subset of a topological vector space we define the classes of maps $\hat{K}(X,Y)$ and $K(X,Y)$ as follows:

$$T \in \hat{K}(X,Y) \Leftrightarrow T \text{ is u.s.c. with compact values;}$$

$$T \in K(X,Y) \Leftrightarrow T \in \hat{K}(X,Y) \text{ and } T(x) \neq \phi \text{ for each } x \in X .$$

Throughout this paper, we assume that any topological space is Hausdorff.

3. Main result

The starting point is the following fixed point theorem:

Theorem 1. Let *X*, *Y* be two nonempty compact convex sets, each in a locally convex topological vector space. Then for every two maps $S \in K(X,Y), T \in K(Y,X)$, the composition $T \circ S$ has a fixed point.

Proof. Consider the diagram

$$X \times Y \xrightarrow{p} Y \times X \overset{T \times S}{\multimap} X \times Y$$

where $p(x,y) = (y,x)$ and $(T \times S)(y,x) = T(y) \times S(x)$.It is easy to see that $[T \times S] \circ p \in K(X \times Y, X \times Y)$, hence by the Fan-Glicksberg fixed point theorem [5, 8], the map $[T \times S] \circ p$ has a fixed point. Therefore for some $(x_0, y_0) \in X \times Y$ we have $(x_0, y_0) \in (T \times S)(y_0, x_0)$. Then $x_0 \in Ty_0, y_0 \in Sx_0$ and consequently $x_0 \in (T \circ S)(x_0)$.

The previous result is a particular case of Theorem 4 in [12]. On the other hand since any fixed point for $T \circ S$ is a coincidence point for the maps *T* and *S*, Theorem 1 is equivalent with Theorem 4 in [9].

The next two results are direct consequences of Theorem 1.

Theorem 2. Let *X*, *Y* be two nonempty compact convex sets, each in a locally convex topological space, $S : X \to Y$ a map with nonempty values and open fibers and $T \in K(X \times Y)$. Then $T \circ S$ has a fixed point.

Proof. It is well known that under the hypothesis of our theorem *S* admits a continuous selection (see Ben-El-Mechaiekk, Deguire and Granas [2, 3]). In other words there is a continuous function $s : X \to Y$ such that $s(x) \in S(x)$ for all $x \in X$.

Since $s \in K(X,Y)$, by Theorem 1 there exists $x_0 \in X$ such that $x_0 \in (T \circ s)(x_0)$. Obviously x_0 is a fixed point for $T \circ S$.

Theorem 3. Let *X*, *Y* be two nonempty compact convex sets, each in a locally convex topological vector space and *M*, *N* be two open subsets of *X*×*Y* such that $M \cup N = X \times Y$. Suppose that the following conditions are satisfied:

(i) For each $x \in X, \{y \in Y : (x, y) \notin M\}$ is convex;
(ii) For each $y \in X, \{y \in Y : (x, y) \notin N\}$ is convex.
Then at least one of the following assertion holds:
(a) There exists a point $x_0 \in X$ such that $\{x_0\} \times Y \subset M$.
(b) There exists a point $y_0 \in Y$ such that $X \times \{y_0\} \subset N$.

Proof. Let $M' = (X \times Y) \setminus M$ and $N' = (X \times Y) \setminus N$. Define $S : X \to Y, T : Y \to X$ by putting

$$S(x) = \{y \in Y : (x, y) \in M'\},\ T(y) = \{x \in X : (x, y) \in N'\}$$

Since M' is closed in $X \times Y$, each $S(x)$ is closed in Y and the graph of S is closed in $X \times Y$. Hence S is u.s.c. and by (ii) it follows that $S \in \hat{K}(X,Y)$.
Similarly we can prove that $T \in \hat{K}(X,Y)$.
Suppose that both assertions (i) and (ii) are not true. Then for each $x \in X$ there exists $y \in Y$ such that $(x, y) \in M'$, that is $S \in K(X,Y)$ and similarly $T \in K(X,Y)$. By Theorem 1, $T \circ S$ has a fixed point, or equivalently, there exists $(x_0, y_0) \in X \times Y$ such that $y_0 \in S(x_0)$ and $x_0 \in T(y_0)$. Then, $(x_0, y_0) \in M' \cap N'$ which contradicts $M \cup N = X \times Y$.

Corollary 4. Let *X*, *Y* be two nonempty compact convex sets, each in a locally convex vector topological space and *N* be an open subset of *X* × *Y* satisfying:

(i) There exists a map $T \in K(X,Y)$ such that $graphT \in N$.
(ii) For each $y \in Y, \{x \in X : (x, y) \notin N\}$ is convex.
Then there exists a point $y_0 \in Y$ such that $X \times \{y_0\} \subset N$.

Proof. Consider the set

$$M = X \times Y \setminus graphT$$

Since $T \in K(X,Y)$ it readily follows that:

$$\begin{cases} M \text{ is an open subset of } X \times Y; \\ \text{for each } x \in X,\ \{y \in Y : (x,y) \notin M\} \text{ is convex;} \\ \text{for each } x \in X,\ \{x\} \times Y \not\subset M. \end{cases}$$

Moreover $M \cup N = X \times Y$. The conclusion follows from Theorem 3.

Corollary 5. Let X be a nonempty compact convex subset of a locally convex vector topological space and M be an open subset of $X \times X$ satisfying:

(i) $\Delta = \{(x,x) : x \in M\} \subset M$

(ii) For each $x \in X, \{y \in X : (x,y) \notin M\}$ is convex.

Then there exists a point $x_0 \in X$ such that $\{x_0\} \times X \subset M$.

Proof. Apply Theorem 3 in the case $Y = X, N = X \times X \setminus \Delta$ and observe that the assertion (b) in the conclusion of this theorem cannot take place.

Theorem 6. Let X, Y, M, N be as in Theorem 3. Suppose that for each $x \in X$ there exists an open subset (possibly empty) O_x of Y such that:

(iii) For each $x \in X, O_x \subset \{y \in Y : (x,y) \in N\}$.

(iv) $\cup_{x \in X}\ x = Y$.

Then there exists $x_0 \in X$ such that $\{x_0\} \times Y \subset M$.

Proof. It suffices to prove that under conditions (iii) and (iv) the assertion (b) of the conclusion of Theorem 3 does not hold.

Since Y is compact there exists a finite set $A = \{x_1, x_2, ..., x_n\} \subset X$ such that $Y = \cup_{i=1}^{n} O_{x_i}$. Let $\{\alpha_i : 1 \le i \le n\}$ be a continuous partition of unity subordinated to the open covering $\{O_{x_i} : 1 \le i \le n\}$ of the compact Y, that is, for each $i, \alpha_i : Y \to [0,1]$ is continuous;

$$\begin{cases} \alpha_i(y) > 0 \Rightarrow y \in O_{\boldsymbol{x}}; \\ \sum_{i=1}^{n} \alpha_i = 1 \text{ for each } y \in Y. \end{cases}$$

Define a continuous function $p : X \to Y$ by

$$p(x) = \sum_{i=1}^{n} \alpha_i(y) x_i.$$

Let $J(y) = \{x_i \in A : \alpha_i(y) > 0\}$. Then $p(y) \in \operatorname{conv}\{x_i : i \in J(y)\}$. For each $x_i \in J(y)$ we have $y \in O_x$, hence by (iii), $(x_i, y) \notin N$. Since the sets $\{x \in X : (x, y) \notin N\}$ are convex (see condition (ii) in Theorem 3) we infer $(p(y), y) \notin N$ for each $y \in Y$, hence the assertion (b) of the conclusion of Theorem 3 does not hold.

Theorem 7. Let *X*, *Y* be two nonempty compact convex sets each in a locally convex vector topological space and $f, g : X \times Y \to IR$ two functions satisfying:

(i) $f \leq g$;

(ii) f is upper semicontinuous and g is lower semicontinuous on $X \times Y$.

(iii) For each $x \in X$, $f(x,\cdot)$ is quasiconcave on Y.

(iv) For each $y \in Y, g(\cdot, y)$ is quasiconcave on *X*.

Then, given any $\alpha, \beta \in IR, \alpha < \beta$, at least one of the following assertions holds:

(a) There exists $x_0 \in X$ such that $f(x_0, y) < \alpha$ for each $y \in Y$.

(b) There exists $y_0 \in Y$ such that $f(x, y_0) > \beta$ for each $x \in X$.

Proof. Apply Theorem 3 to the sets:

$$M = \{(x, y) \in X \times Y : f(x, y) < \alpha\},$$
$$N = \{(x, y) \in X \times Y : g(x, y) > \beta\}.$$

From the hypothesis (i) – (iv) it follows readily that *M*, *N* are open in $X \times Y$, $M \cup N = X \times Y$ and assumptions (i) – (iii) of Theorem 3 are verified. The desired result follows now from Theorem 3.

It would be of some interest to compare the next minimax inequality with the generalizations of the Neumann minimax theorem obtained by Simons [14] and Nikaido [13].

Corollary 8. Under the hypotheses of Theorem 7 the following inequality holds:

$$\inf_{x \in X} \max_{y \in Y} f(x, y) \leq \sup_{y \in Y} \min_{x \in X} g(x, y).$$

Proof. First let us observe that if f is upper semicontinuous on $X \times Y$, then for each $x \in X$, $f(x,\cdot)$ is also an upper semicontinuous function of y on Y and therefore its maximum $\max_{y\in Y} f(x,y)$ on the compact set Y exists. Similarly $\inf_{x\in X} g(x,y)$ can be replaced by $\min_{x\in X} g(x,y)$.

Suppose the conclusion were false and chose two real numbers α, β such that

$$\sup_{y\in Y} \min_{x\in X} g(x,y) < \beta < \alpha < \inf_{x\in X} \max_{y\in Y} f(x,y).$$

We prove that neither the assertion (a) nor the assertion (b) of the conclusion of Theorem 7 cannot take place.

If (a) happens, then

$$\inf_{x\in X} \max_{y\in Y} f(x,y) \le \max_{y\in Y} f(x_0,y);\ \text{a contradiction.}$$

If (b) happens, then

$$\sup_{y\in Y} \min_{x\in X} g(x,y) \ge \min_{x\in X} g(x,y_0);\ \text{a contradiction again.}$$

The origine of our two last results goes back to Fan's minimax inequalities [6]. Close results have been obtained by Allen [1], Granas and Liu [9], Fan [7] and Ha [10].

Theorem 9. Let *X*, *Y*, *f*, *g* be as in Theorem 7. If $T : X \to Y$ is a map with nonempty values, then the following inequality holds:

$$\inf_{y\in T(x)} f(x,y) \le \sup_{y\in Y} \min_{x\in X} g(x,y).$$

Proof. We may assume that $\inf_{y\in T(x)} f(x,y) > -\infty$. Apply Theorem 7 in the case $\alpha = \inf_{y\in T(x)} f(x,y), \beta = \inf_{y\in T(x)} f(x,y) - \varepsilon$ where $\varepsilon > 0$ is arbitrarly fixed. Since the values of T are nonempty, the assertion (a) of the conclusion of Theorem 7 cannot take place. It follows that there exists $y_0 \in Y$ such that

$$\min_{x\in X} g(x,y_0) > \inf_{y\in T(x)} f(x,y) - \varepsilon.$$

Clearly this implies the desired minimax inequality.

Corollary 10. Let *X* be a nonempty compact subset of a locally convex topological vector space and $f, g : X \times X \to R$ two functions satisfying:

(i) $f \le g$.

(ii) *f* is upper semicontinuous and *g* is lower semicontinuous on *X* ×*X*.

(iii) For each $x \in X$, $f(x,\cdot)$ is quasiconcave on Y.
(iv) For each $y \in Y, g(\cdot, y)$ is quasiconcave on X.
Then we have

$$\inf_{x \in X} f(x,x) \leq \sup_{y \in Y} \min_{x \in X} g(x,y)$$

Proof. Apply Theorem 9 with $X = Y$, $T(x) = \{x\}$.

References

[1] G. Allen, Variational inequalities, complementary problems, and duality theorems, J. Math. Anal. Appl. 58 (1977), 1-10.
[2] H.Ben-El-Mechaiekk, P. Deguire and, A. Granas, Point fixes et coincidences pour les fonctions multivoques (applications de Ky Fan), C. R. Acad. Sci. Paris 295 (1982), 337-340.
[3] H.Ben-El-Mechaiekk, P. Deguire and, A. Granas, Point fixes et coincidences pour les fonctions multivoques II (Applications de type C et C*), C. R. Acad. Sci. Paris 295 (1982), 381-384.
[4] H. F. Bohnenblust and S. Karlin, On a theoreme of Ville, in: Contributions to the Theory of Games, Vol. 24, Ann. Of Math. Studies, Princeton University Press, 1950, pp. 155-160.
[5] K. Fan, Fixed point and minimax theorems in locally convex topological linear spaces, Proc. Nat. Acad. Sci. U.S.A. 38 (1952), 131-136.
[6] F. Fan, A minimax inequality and its applications, Inequality III (O. Shisha, ed.) Academic Press, 1972, pp.103-113.
[7] K. Fan, Some properties of convex sets related to fixed point theorems, Math. Ann. 266 (1984), 519-537.
[8] I. L. Glicksberg, A further generalization of the Kakutani fixed point theorem with applications to Nash equilibrium points, Proc. Amer. Math. Soc. 3 (1952), 170-174.
[9] A. Granas and F. C. Liu, Coincidences for set-valued maps and minimax inequalities,
J. Math. Pures Appl. 65 (1986), 119-148.
[10] C. W. Ha, On a minimax inequality of Ky Fan, Proc. Am. Math. Soc. 99 (1987), 680-682.
[11] S. Kakutani, A generalization of Brouwer's fixed point theorem, Duke Math. J. 8 (1941), 457-459.
[12] M. Lassonde, Fixed points for Kakutani factorizable multifunctions, J. Math. Anal. Appl. 152 (1990), 46-60.
[13] H. Nikaido, On von Neumann's minimax theorem, Pacific J. Math. 4 (1954), 65-72.

[14] S. Simons, Two-functions minimax theorems and variational inequalities for functions on compact and noncompact sets with some comments on fixed-point theorems, in Proc. Symp. Pure Math. (F. E. Brouwer, ed.) vol. 45, Amer. Math. Soc., Providence, Rhode Island, 1986, pp. 377-392.

Authors:
Mircea Balaj and Daniel Erzse
Department of Mathematics, University of Oradea, Oradea, ROMANIA
E-mail address: mbalaj@uoradea.ro; derzse@uoradea.ro