AN INEQUALITY IN EUCLIDEAN SPACES

by
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Abstract. In this paper, we establish the following inequality:

$$2\prod_{i=1}^{n}\langle x_i ; y_i\rangle \le \prod_{i=1}^{n}\|x_i\|^2 + \prod_{i=1}^{n}\|y_i\|^2$$

where $\langle \bullet;\bullet\rangle$ is a positive symmetric bilinear form defined over a vectorial space E; and ||·|| the associated norm.

Let E be a vectorial space endowed with a positive symmetric bilinear form denoted by $\langle \bullet;\bullet\rangle$: One can endow $E\otimes\ldots\otimes E = E^{\otimes n}$ with a positive symmetric bilinear form (associated to $\langle \bullet;\bullet\rangle$)denoted by $\langle (\bullet;\bullet)\rangle$ and defined by:

$$\langle (x_1 \otimes x_2 \otimes \ldots \otimes x_n ; y_1 \otimes y_2 \otimes \ldots \otimes y_n)\rangle = \prod_{i=1}^{n}\langle x_i ; y_i\rangle$$

Theorem

$$2\prod_{i=1}^{n}\langle x_i ; y_i\rangle \le \prod_{i=1}^{n}\|x_i\|^2 + \prod_{i=1}^{n}\|y_i\|^2$$

Proof.

For ® = $x_1 \otimes \ldots \otimes x_n - y_1 \ldots \otimes y_n$ we obtain:

$$<(®;®)> = \langle (x_1 \otimes \ldots \otimes x_n - y_1 \otimes \ldots \otimes y_n ; x_1 \otimes \ldots \otimes x_n - y_1 \ldots \otimes y_n)\rangle$$
$$= \langle (x_1 \otimes \ldots \otimes x_n ; x_1 \otimes \ldots \otimes x_n)\rangle - \langle (x_1 \otimes \ldots \otimes x_n ; y_1 \otimes \ldots \otimes y_n)\rangle$$
$$- \langle (y_1 \otimes \ldots \otimes y_n ; x_1 \otimes \ldots \otimes x_n)\rangle + \langle (y_1 \otimes \ldots \otimes y_n ; y_1 \otimes \ldots \otimes y_n)\rangle$$

$$=\prod_{i=1}^{n}\langle x_i;x_i\rangle-2\prod_{i=1}^{n}\langle x_i;y_i\rangle+\prod_{i=1}^{n}\langle y_i;y_i\rangle$$

and the result follows from <(®;®)> ≥0:

Remark 1

1. For n =2; $x_1 = y_2 = x$ and $x_2 = y_1 = y$, we have the Cauchy-Schwarz inequality, which was established in [1].

2. For $x_i = x$ and $y_i = y$, i=1;…;n, we have:

$$\langle x;y\rangle^n \le \| x_i \|^{2n} + \| y_i \|^{2n}$$

Remark 2

If we consider the inner product of E over | the inequality becomes

$$2\Re e\left(\prod_{i=1}^{n}\langle x_i;y_i\rangle\right) \le \prod_{i=1}^{n}\|x_i\|^2 + \prod_{i=1}^{n}\|y_i\|^2$$

For the first case of remark 1

$$\Re e(\langle x;y\rangle\langle y;x\rangle) = \Re e(\langle x;y\rangle\overline{\langle x;y\rangle}) = \Re e\left(|\langle x;y\rangle|^2\right) = |\langle x;y\rangle|^2$$

which gives the Cauchy-Schwarz inequality when E is a vectorial space over |:

References

[1] Bouzar C. *Une identitè dans les espaces prèhilbertiens*. 2ième colloque d'analyse fonctionnelle et applications, 17-19 novembre1997, Sidi Bel Abbes, ALGERIE.
[2] Mitrinovitch P.S, Vasic P.M. *Analytic Inequalities*. Springer Verlag, 1970.

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