SOLVING A BILOCAL LINEAR SINGLE PERTURBATED PROBLEM

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Abstract. This paper presents algorithms for solving a boundary layer bilocal linear single perturbated problem, which appears when we want to describe certain flows in the fluids mechanics. We can obtain very good results when solving this problem through approximation methods, results that are compared in the end of this paper to the numerical results.

Key words: outer expansion, inner expansion, the matching.

As we all know from the specialized literature. There are many methods to determine a uniform solution for the single perturbated problems attached to some bilocal problems, methods that are technical but most of the time laborious, requiring expensive calculation in order to obtain an approximacy of a precise exactly. There, for the study of such single perturbated problems attached to some ordinary differential equation, of first order can be performed in very good conditions using approximation methods. As a consequence to these problems we look for asimptotic solutions using the method of the matching asymptotic expansion and the multiple scales method. In both cases the results are compared to the numerical results.

Let's consider the following bilocal linear problem:

$$\varepsilon y'' + y' + y = 0 \tag{1a}$$

$$y(0) = \alpha, \quad y(1) = \beta \tag{1b}$$

for which we determine a first order uniform expansion through the two mentioned approximation methods, expansions compared afterwards to (the precise solution) exact solution of the problem

Exact solution of the problem

We shall look for the solution for the homogeneous equation (1a) under the following exponential form y = exp (sx). Then (1a) will have the next characteristic equation:

$$\varepsilon s^2 + s + 1 = 0$$

which has the following solutions

$$s_1 = \frac{-1+\sqrt{1-4\varepsilon}}{2\varepsilon} \qquad si \qquad s_2 = \frac{-1-\sqrt{1-4\varepsilon}}{2\varepsilon}$$

There for the general solution of the (1a) equation is:

$$y = c_1\, e^{s_1 x} + c_2\, e^{s_2 x} \tag{2a}$$

Substituting (2a) in (1b) we shall obtain the following system:

$$\begin{cases} c_1 + c_2 = \alpha \\ c_1 e^{s_1} + c_2 e^{s_2} = \beta \end{cases}$$

which has the next solutions:

$$c_1 = \frac{\alpha e^{s_2} - \beta}{e^{s_2} - e^{s_1}},$$

$$c_2 = \frac{\beta e^{s_2} - \alpha}{e^{s_2} - e^{s_1}}$$

So, the solution of the (1a) homogeneous equation is:

$$y = \frac{\left(\alpha e^{s_2} - \beta\right) e^{s_1 x} + \left(\beta e^{s_2} - \alpha\right) e^{s_2 x}}{e^{s_2} - e^{s_1}} \tag{2b}$$

The matching asymptotic expansions method

The outer expansion. We choose the first-order outer-expansion under the next form:

$$y^{o} = y_0(x) + \dots \tag{3}$$

Since the small parameter multiplies the largest derivative and the position of the boundary layer depends on the y ` coefficient which is 1 and it is positive, the boundary layer is in x = 0 (in the vicinity of the origin). So the outer expansion has to satisfy the second condition, namely condition (1b). Substituting (3) in (1a) with y(1)=β and equalising coefficients of the same power of ε, we obtain:

$$y_0' + y_0 = 0, \qquad y_0(1) = \beta$$

which has the next solution:

$$y_0 = \beta\, e^{1-x}$$

Therefore, the outer expantion will be written under the following form:

$$y^{o} = \beta\, e^{1-x} + \dots \tag{4}$$

The inner expansion. In order to be able to study y behaviour in the boundary layer, we have to introduce the modified transformation:

$$\xi = \frac{x}{\varepsilon^{\lambda}}, \quad \lambda > 0$$

Under the terms of the modified variable, equation (1a) can be written like this:

$$\varepsilon^{1-2\lambda} \frac{d^2 y^i}{d\xi^2} + \varepsilon^{-\lambda} \frac{dy^i}{d\xi} + y^i = 0 \tag{5}$$

of which minimum degenerated form is:

$$\frac{d^2 y^i}{d\xi^2} + \frac{dy^i}{d\xi} - = 0$$

subject to λ = 1, for ε → 0, where y^i is a note for y belonging to the boundary layer. In this way we chose the inner expansion under the form:

$$y^i = Y_0(\xi) + \dots, \qquad \xi = \frac{x}{\varepsilon} \tag{6}$$

The inner expansion has to satisfy the first boundary condition in (1b). Since x = 0 corresponds to
$\xi = 0$,

$$y^i(0) = \alpha \tag{7}$$

Substituting (6) in (5), with $\lambda = 1$, and in (7) and equalising the same powers of ε, we obtain the first approximation under the form:

$$Y_0'' + Y_0' = 0, \qquad Y_0(0) = \alpha$$

which has the solution:

$$Y_0 = \alpha - c_1 + c_1 e^{-\xi}$$

Therefore:

$$y^i = \alpha - c_1 + c_1 e^{-\xi} + \dots \tag{8}$$

The matching. Next, we match the inner expansion to the outer expansion, term by term, so we obtain:

- The first term of the outer expansion: $y \sim \beta e^{1-x}$
- we rewrite it in the inner variable: $= \beta e^{1-\varepsilon\xi}$
- we develop for the small ε $= \beta e\,(1 - \varepsilon\xi + \dots)$
- the first term of the inner expansion $= \beta e$ (9a)
- The first term of the inner expansion: $y \sim \alpha - c_1 + c_1 e^{-\xi}$
- we rewrite it in the inner variable: $= \alpha - c_1 + c_1 e^{-x/\varepsilon}$
- we develop for the small ε $= \alpha - c_1 + \text{EST}$
- the first term of the inner expansion $= \alpha - c_1$ (9b)

Equalising (9a) with (9b), after the main matching we obtain

$$\alpha - c_1 = \beta e$$

so

$$c_1 = \alpha - \beta e$$

and the inner expansion is written:

$$y^i = \beta e + (\alpha - \beta e)e^{-\xi} + \dots$$

The relation will give the form of the uniform expansion

$$y^c = y^0 + y^i - \left(y^0\right)^i = \beta e^{1-x} + \beta e + (\alpha - \beta e)e^{-\xi} - \xi e + \dots$$

or

$$y^c = \beta e^{1-x} + (\alpha - \beta e)e^{-\xi} \tag{10a}$$

We compared (10a) to the exact solution (1b) and we note that:

$$s_1 = \frac{-1 + 1 - 2\varepsilon + \dots}{2\varepsilon} = -1 + \dots$$

and

$$s_2 = \frac{-1 - 1 + 2\varepsilon + \dots}{2\varepsilon} = -\frac{1}{\varepsilon} + 1 + \dots$$

Therefore exp (s_2) is a much smaller exponent, if compared to exp (s_1), so negligible, therefor (2b) is written:

$$y^e = \beta e^{1-x} + (\alpha - \beta e)e^{x - x/\varepsilon} + \dots \tag{10b}$$

and we have

$$y^e - y^c = (\alpha - \beta e)e^{-x/\varepsilon}\left(e^x - 1\right) + \dots$$

which is of the O(ε) order in the inner region x = O(ε) and of a small exponent in the outer region.

The multiple scales methods

We shall choose the first-order outer expansion under the form:

$$y(x, \varepsilon) = y_0(\xi, \eta) + \varepsilon y_1(\xi, \eta) + \dots \tag{11a}$$

where

$$\eta = x \qquad \text{and} \qquad \xi = \frac{x}{\varepsilon}$$

then, the first and second order partial derivates of y will be:

$$\frac{dy}{dx} = \frac{1}{\varepsilon}\frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta}, \quad \frac{d^2 y}{dx^2} = \frac{1}{\varepsilon^2}\frac{\partial^2 y}{\partial \xi^2} + \frac{2}{\varepsilon}\frac{\partial^2 y}{\partial \xi \partial \eta} + \frac{\partial^2 y}{\partial \eta^2} \tag{11b}$$

and equation (1a) becomes:

$$\frac{1}{\varepsilon}\frac{\partial^2 y}{\partial \xi^2} + 2\frac{\partial^2 y}{\partial \xi \partial \eta} + \varepsilon\frac{\partial^2 y}{\partial \eta^2} + \frac{1}{\varepsilon}\frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta} + y = 0 \tag{12}$$

Substituting (11a) in (20) and equalising the coefficients of the same powers of ε, we obtain:

$$\frac{\partial^2 y_0}{\partial \xi^2} + \frac{\partial y_0}{\partial \xi} = 0 \tag{13}$$

$$\frac{\partial^2 y_1}{\partial \xi^2} + \frac{\partial y_1}{\partial \xi} = -2\frac{\partial^2 y_0}{\partial \xi \partial \eta} - \frac{\partial y_0}{\partial \xi} - y_0 \tag{14}$$

The general solution of equation (13) must be written under the form:

$$y_0 = A(\eta) + B(\eta)e^{-\xi} \tag{15}$$

where A and B are functions that will be determined out of a line of approximates, given by (14). The equation (14) becomes

$$\frac{\partial^2 y_1}{\partial \xi^2} + \frac{\partial y_1}{\partial \xi} = -(A' + A) + (B' - B)e^{-\xi}$$

The relation given by y_1 / y_0, for any ξ, must be limited, Therefore we have:

$$A' + A = 0, \qquad B' - B = 0$$

from where we obtain the general solution under the form:

$$A = ae^{-\eta}, \qquad B = be^{-\xi} \tag{16}$$

A and B being constant. Substituting (16) in (15), and the result in (11a) we obtain:

$$y = ae^{-\eta} + be^{\eta-\xi} + \dots$$
or
$$y = ae^{-x} + be^{x-x/\varepsilon} + \dots \tag{17}$$

Substituting (17) in (1b) we obtain the system

$$\begin{cases} a + b = \alpha \\ ae^{-1} + be^{1-1/\varepsilon} = \beta \end{cases}$$

which has the solution

$$a = \beta e + EST$$
and
$$b = \alpha - \beta e$$

Then equation (17) can be written under the next form:

$$y = \beta e^{1-x} + (\alpha - \beta e)e^{x-x/\varepsilon} + \dots \tag{18}$$

We can notice that equation (18) is in perfect concordance with equation (10b), meaning, with the numerical solution of the problem.

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