CONTINUOUS SELECTIONS OF SOLUTION SETS TO SECOND ORDER EVOLUTION EQUATIONS

by
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Abstract. We prove the existence of a continuous selection of the multivalued map $(\xi,\eta) \rightarrow$ $A_F(\xi,\eta)$, where $A_F(\xi,\eta)$ is the set of all mild solutions of the Cauchy problem

$$x'' \in Ax + F(t, x) \ , \ x(0) = \xi \ , x(0) = \eta$$

assuming that F is Lipschitzian with respect to x and A is the infinitesimal generator of a strongly cosine family of linear operators on a Banach space E.
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1. Introduction

The existence of continuous selections of solution sets to the Cauchy problem

$$x' \in F(t, x) \ , \ x(0) = \xi,$$

where F(., .) is a Lipschitzian multifunction with respect to x, has been proved first by Cellina in [4]; some extensions have been treated in [5] and [6]. In [12] similar results are proved for Cauchy problem

$$x \in Ax + F(t, x) \ , \ x(0) = \xi,$$

where F (. , .) is a Lipschitzian multifunction with respect to x, with nonempty closed values and -A is a maximal monotone map (resp. A is the infinitesimal generator of a C_0-semigroup). For another the existence results for second order differential equations see [2], [10] and [14].

The purpose of the present paper is to prove the existence of a continuous mapping $(\xi,\eta) \rightarrow x(.,\xi,\eta)$, is a solution of the Cauchy problem

$$x'' \in Ax + F(t, x) \ , \ x(0) = \xi \ , x(0) = \eta$$

under the assumption that F (. , .) is a Lipschitzian multifunction with respect to x, with nonempty closed values and A is the infinitesimal generator of a cosine family of operators on a separable Banach space.

2. Preliminaries

Let T > 0 and E be a separable Banach space with norm || . ||. For x $\in$ E and A, B any two closed subset of E we define the distance from x to A by d(x,A) := inf {||x

– y|| ; y ∈ A }and the Hausdorff-Pompeiu distance from A to B by h(A,B) := inf { t > 0 ; A ⊂ B + tU , B ⊂ A +tU }, where U := {x ∈ E ; ||x|| ≤ 1 }. For any subset A ⊂ E, we denote by cl(A) the closure of A. Denote by L the σ-field of Lebesgue measurable subset of [0, T] and by *B*(E) the family of all Borel subset of E.

Let $L^1([0, T], E)$ be the Banach space of all absolutely continuous functions x : [0 ,T] → E endowed with norm $\|x\|_1 := \int_0^T \| x(t) \| dt$, C ([0 ,T], E) the Banach space of all continuous functions x :[0, T] → E with the norm $\| x \|_\infty := \sup\{ x(t) ; t \in [0, T]\}$and $C^1([0, T], E)$ the Banach space of all functions x : [0, T] → E which are continuous differentiable with the norm $\| x \|_{\infty,1} := \max \{ \| x \|_\infty , \| x'\|_\infty \}$.

Recall that a subset K of L^1 ([0 ,T], E) is said to be decomposable ([8]) if for every u, v ∈ K and A ⊂ L we have $u_{\aleph A} + v_{\aleph[0,T]\setminus A} \in K$, where $\aleph A$ is characteristic function of A. We denote by D the family of all decomposable closed nonempty subset of L^1 ([0 ,T], E).

Let G : [0, T] → 2^E be a multifunction. G is called L-measurable if for every closed subset A of E the set {t ∈ [0 ,T] ; G (t)∩ A ≠ Ø} ∈ L. A function g :[0, T] → E is called a selection of G if g(t) ∈ G (t) for all t ∈ [0, T]. Let S be a separable metric space. A multivalued mapping G : S → 2^E is said to be lower semicontinuous (l.s.c.) if for every closed subset A of E the set {s ∈ S ; G(s) ⊂ A}is closed in E.

We say that the family { C(t) ; t ∈ R } in the space B(E) of bounded linear operators is a strongly continuous cosine family if:

(i) C(0) = I (I is the identity operator in E),

(ii) t →C(t)x is strongly continuous for each fixed x ∈ E,

(iii) C(t + s) + C(t – s) = 2C(t)C(s) for all t, s ∈ R.

The strongly continuous sine family { S(t) ; t ∈ R }, associated to the given strongly continuous cosine family {C(t) ; t ∈ R }, is defined by

$$S(t)x := \int_0^t C(s)xds \ , \ x \in E , t \in R .$$

The infinitesimal generator A : E → E of a cosine family {C(t) ; t ∈ R } is defined by

$$Ax := \frac{d^2}{dt^2} C(0)x.$$

We denote by D(A) the domain of A, that is

D(A) := { x ∈ E ; C(t)x is twice continuous differentiable }

and by E_0 the set

E_0 := { x ∈ E ; C(t)x is once continuous differentiable }.

If $x \in E$ then for any $t, s \in R$ we have that $\int_s^t S(\tau)x d\tau \in E_0$ and $S(t)x \in E_0$. If $x \in E_0$ then $C(t)x \in E_0$ and $S(t)x \in D(A)$.

For more details on strongly continuous cosine and sine family, we refer to the book [7] and the paper [9, 11, 13, 14].

Consider a multifunction $F : [0, T] \times E \to 2^E$ satisfying the following assumptions:

(H_1) F is $L \otimes B(E)$-measurable,

(H_2) there exists $k(\,.\,) \in L^1([0,T], R_+)$ such that $h(F(t, x), F(t, y)) \leq k(t)\|x - y\|$, for all $x,y \in E$, a.e. $t \in [0,T]$,

(H_3) there exist $\beta(\,.\,) \in L^1([0,T], R_+)$ such that $d(0, F(t, 0)) \leq \beta(t)$ a.e. $t \in [0,T]$.

For such a multifunction F and ξ, η in E we consider the Cauchy problem

$$x'' \in Ax + F(t, x),\ x(0) = \xi,\ x(0) = \eta \qquad (2.1)$$

Definition 2.1. A function $x(\,.;\xi,\eta) : [0,T] \to E$ is said to be a mild solution of the Cauchy problem (CP) if there exists $f(\,.;\xi,\eta) \in L^1([0,T], E)$ such that:

(i) $f(\,.;\xi,\eta) \in F(t, x(t;\xi,\eta))$ for almost all $t \in [0,T]$,

(ii) $x(t;\xi,\eta) = C(t)\xi + S(t)\eta + \int_0^t C(t-s)f(t;\xi,\eta)ds$

We denote by $A_F(\xi,\eta)$ the set of all mild solutions of Cauchy problem (CP).

The main result is the following:

Theorem 2.2. Let $F : [0, T] \times E \to 2^E$ satisfy (H_1) – (H_3). Then there exists $x(\,.\,;\,.\,,\,.\,) : [0, T] \times E \times E \to E$ such that:

(a) $x(\,.\,;\xi,\eta) \in A_F(\xi,\eta)$ it for all $(\xi,\eta) \in E_0 \times E$,

(b) $(\xi,\eta) \to x(\,.\,;\xi,\eta)$ is continuous from $E_0 \times E$ into $C^1([0, T], E)$.

3. Proof of the main result

Let S be a separable metric space. To prove the main result we shall use the following lemmas.

Lemma 3.1. [6,Proposition 2.1] Let $\widetilde{F}$: [0 ,T] × S $\to 2^E$ be L ⊗ *B*(E)-measurable and such that $\widetilde{F}$(t, .) is l.s.c. for each t ∈ [0, T]. Then mapping $\widetilde{G}$:S $\to 2^{L^1([0,T],E)}$ given by

$$\widetilde{G}(s) := \{v \in L^1([0, T], E) ; v(t) \in \widetilde{F}(t, s) \text{ a.e. } t \in [0, T]\}$$

is l.s.c. with closed nonempty and decomposable values if and only if there exists a continuous mapping β : S → L_1([0 ,T], E) such that for all s ∈ S we have d(0, $\widetilde{F}$(t, s)) ≤ β (s)(t), for a.e. t ∈ [0, T].

Lemma 3.2. [6, Proposition 2.2] Let G : S → D be a l.s.c. multifunction and let φ : S → L_1([0 ,T], E) and ψ : S → L_1([0 ,T], R) be continuous maps. If for every s ∈ S the set

$$H(s) := cl\{v \in G(s) ; \|u(t) - \varphi(s)(t)\| < \psi(s)(t), \text{ a.e. } t \in [0, T] \} \quad (3.1)$$

is nonempty then the multifunction mapping s → H(s) defined by (3.1) admits a continuous selections.

Proof of the theorem. Let ε > 0 be fixed, M := max { t ∈ [0, T] sup||C(t)|| , t ∈ [0, T] sup||C'(t)|| }and, for n ∈ N, let $\varepsilon_n := \varepsilon/2^{n+1}$.

For each (ξ, η) ∈ E_0 × E , define $x_0(. ; \xi, \eta) : [0 ,T] \to E$ by $x_0(t ; \xi, \eta) := C(t)\xi + S(t)\eta$.

Since

$\| x_0(t; \xi_1, \eta_1) - x_0(t; \xi_2, \eta_2) \| = \| C(t)(\xi_1 - \xi_2) + S(t)(\eta_1 - \eta_2) \| \le M \| (\xi_1, \eta_1) - (\xi_2, \eta_2) \|$

and

$\| x'_0(t; \xi_1, \eta_1) - x'_0(t; \xi_2, \eta_2) \| = \| C'(t)(\xi_1 - \xi_2) + S'(t)(\eta_1 - \eta_2) \| \le M \| (\xi_1, \eta_1) - (\xi_2, \eta_2) \|$

we have that

$$\| x_0(t; \xi_1, \eta_1) - x_0(t; \xi_2, \eta_2) \| \le M \| (\xi_1, \eta_1) - (\xi_2, \eta_2) \| ,$$

hence (ξ, η) → x (t ; ξ, η) is Lipschitzian and therefore continuous. Dene α(ξ, η) : [0 ,T] → R by $\alpha(\xi, \eta)(t) := \beta(t) + k(t)\|x_0(t; \xi, \eta)\|$ and remark that α(., .) is Lipschitzian, hence continuous, from E_0 × E into L_1([0 ,T], R). Moreover, by (H_1) and (H_2) we have

$$\begin{aligned} d(0, F(t, x_0(t; \xi, \eta))) &\le d(0, F(t, 0)) + h(F(t,0), F(t, x_0(t; \xi, \eta))) \\ &\le \alpha(\xi, \eta)(t). \end{aligned} \quad (3.2)$$

Let $G_0(., .) : E_0 \times E \to 2^{L^1([0,T],E)}$ and $H_0(., .) : E_0 \times E \to 2^{L^1([0,T],E)}$ be defined by

$$G_0(\xi,\eta) := \{v \in L^1([0,T],E)\,;\, v(t) \in F(t, x_0(t;\xi,\eta)) \text{ a.e. } t \in [0,T]\}$$

and

$$H_0(\xi,\eta) := \{v \in G_0(\xi,\eta)\,;\, \|v(t)\| < \alpha(\xi,\eta)(t) + \varepsilon_0 \text{ a.e. } t \in [0,T]\}$$

By (3.2) and Lemma 3.1 it following that $G_0(.,.)$ is l.s.c. from $E_0 \times E$ into D and $H_0(\xi,\eta) \neq \emptyset$, for all $(\xi,\eta) \in E_0 \times E$. Then, by Lemma 3.2, there exists $h_0(.,.) : E_0 \times E \to L^1([0,T],E)$ a continuous selection of $H_0(.,.)$.

Let $m(t) := \int_0^t k(s)ds$ and for $n \geq 1$define $\beta_n(.,.) := E_0 \times E \to L^1([0,T],R)$ by

$$\beta_n(\xi,\eta)(t) = M^n \int_0^t \alpha(\xi,\eta)\frac{[K(t)-K(s)]^{n-1}}{(n-1)!}ds + M^n T(\sum_{i=1}^n \varepsilon_i)\frac{[K(t)]^{n-1}}{(n-1)!}.$$

Set $f_0(t;\xi,\eta) := h_0(\xi,\eta)(t)$ and dene

$$x_0(t;\xi,\eta) := C(t)\xi + S(t)\eta + \int_0^t S(t-s)f_0(s;\xi,\eta)ds,\ t \in [0,T]$$

Then $f_0(t;\xi,\eta) \in F(t, x_0(t;\xi,\eta))$, $\|f_0(t;\xi,\eta)\| \leq \alpha(\xi,\eta)(t) + \varepsilon_0$, a.e. $t \in [0,T]$ and for all $t \in [0,T]$:

$$\|x_1(t;\xi,\eta) - x_0(t;\xi,\eta)\| \leq \int_0^t \|S(t-s)\|\|f_0(s;\xi,\eta)\|ds$$

$$\leq M\int_0^t \|f_0(s;\xi,\eta)\|ds \leq M\int_0^t \alpha(\xi,\eta)(s)ds + \varepsilon_0 MT \leq \beta_1(\xi,\eta)(t)$$

and similarly

$$\|x'_1(t;\xi,\eta) - x'_0(t;\xi,\eta)\| \leq M\int_0^t \alpha(\xi,\eta)(s)ds + \varepsilon_0 M \leq \beta_1(\xi,\eta)(t)$$

Therefore,

$$\|x_1(t;\xi,\eta) - x_0(t;\xi,\eta)\| \leq M(\|\alpha(\xi,\eta)\|_1 + \varepsilon_0 T).$$

We claim that there exist two sequences $(f_n(.;\xi,\eta))_{n\in N}$ and $(x_n(.;\xi,\eta))_{n\in N}$ satisfying for each $n \geq 1$ the following properties:

(i) $(\xi,\eta) \to f_n(.;\xi,\eta)$ is continuous from $E_0 \times E$ into $L^1([0,T],E)$.

(ii) $f_n(.;\xi,\eta) \in F(t, x_n(t;\xi,\eta))$ for all $(\xi,\eta) \in E_0 \times E$, a.e. $t \in [0,T]$,

(iii) $\|f_n(t;\xi,\eta) - f_{n-1}(t;\xi,\eta)\| \leq k(t)\beta_n(\xi,\eta)(t)$, a.e. $t \in [0,T]$,

(iv) $x_n(t;\xi,\eta) = C(t)\xi + S(t)\eta + \int_0^t S(t-s)f_{n-1}(s;\xi,\eta)ds$, $t\in [0,T]$

Suppose we have already constructed $f_1, f_2 \dots f_n$ and $x_1, x_2 \dots x_n$ satisfying (i)-(iv).

Then dene $x_n(.;.,.)$ by

$$x_n(t;\xi,\eta) = C(t)\xi + S(t)\eta + \int_0^t S(t-s)f_{n-1}(s;\xi,\eta)ds \ , t\in [0,T]$$

and remark that (iii) and (iv) implies

$$\| x_{n+1}(t;\xi,\eta) - x_n(t;\xi,\eta)\| \le \int_0^t \| S(t-s)\|\| f_n(s;\xi,\eta) - f_{n-1}(s;\xi,\eta)\| ds$$

$$\le \int_0^t k(s)\beta_n(\xi,\eta)(s)ds = M^{n+1}[\int_0^t \alpha(\xi,\eta)(s)\int_s^t k(\tau)\frac{[m(t)-K(\tau)]^n}{n!}d\tau ds$$

$$+ T(\sum_{i=1}^n \varepsilon_i)\int_0^t \frac{[m(s)]^n}{n!}ds\,]$$

$$= M^{n+1}[\int_0^t \alpha(\xi,\eta)(s)\frac{[m(t)-m(s)]^n}{n!}ds + T(\sum_{i=1}^n \varepsilon_i)\int_0^t \frac{[m(t)]^n}{n!}ds\,]$$

$$\le \beta_{n+1}(\xi,\eta)(t) \quad (3.3)$$

and

$$\| x'_{n+1}(t;\xi,\eta) - x'_n(t;\xi,\eta)\| \le \beta_{n+1}(\xi,\eta)(t). \quad (3.4)$$

Also, by (H_2), we have that

$$d(f_n(t;\xi,\eta), F(t; x_{n+1}(t;\xi,\eta))) \le k(t)\| x_{n+1}(t;\xi,\eta) - x_n(t;\xi,\eta)\|$$
$$\le k(t)\beta_{n+1}(\xi,\eta)(t). \quad (3.5)$$

Let $G_{n+1}(.,.) : E_0 \times E \to 2^{L^1([0,T],E)}$ and $H_{n+1}(.,.) : E_0 \times E \to 2^{L^1([0,T],E)}$ be defined by

$$G_{n+1}(\xi,\eta) := \{v \in L^1([0,T],E) ; v(t) \in F(t, x_n(t;\xi,\eta)) \text{ a.e. } t \in [0,T]\}$$

and similary

$$H_{n+1}(\xi,\eta) := \{v \in G_{n+1}(\xi,\eta) ; \|v(t) - f_n(\xi,\eta)\| < k(t)\beta_{n+1}(\xi,\eta)(t). \text{ a.e. } t \in [0,T]\}$$

By (3.5)and Lemma 3.1 it following that $G_{n+1}(.,.)$ is l.s.c. from $E_0 \times E$ into D and $H_{n+1}(\xi,\eta) \ne \emptyset$, for all $(\xi,\eta) \in E_0 \times E$. Therefore, by Lemma 3.2, there exists $h_{n+1}(.,.) : E_0 \times E \to L^1([0,T],E)$ a continuous selection of $H_{n+1}(.,.)$ and setting $f_{n+1}(t;$

$\xi, \eta) := h_{n+1}(\xi, \eta)(t)$ we obtain that $f_n(.; ., .)$ satisfies the properties (i)-(iii)and by (3.3) it follows

$$\| f_{n+1}(t;\xi,\eta) - f_n(t;\xi,\eta) \|_1 = \int_0^t \| f_n(s;\xi,\eta) - f_{n-1}(s;\xi,\eta) \| ds$$

$$\leq M^n [\int_0^t \alpha(\xi,\eta)(s) \frac{[K(t)-K(s)]^n}{n!} ds + M^n T(\sum_{i=1}^n \varepsilon_i) \frac{[K(t)]^n}{n!}$$

$$\leq \frac{M \| k \|_1}{n!} (\| \alpha(\xi,\eta) \|_1 + \varepsilon_0 T). \qquad (3.6)$$

By (iii) and (3.6) it follows that

$$\| x_{n+1}(.;\xi,\eta) - x_n(.;\xi,\eta) \|_\infty \leq M \| f_n(s;\xi,\eta) - f_{n-1}(s;\xi,\eta) \|_1$$

$$\leq \frac{[M \| k \|_1]^n}{n!} (\| \alpha(\xi,\eta) \|_1 + \varepsilon_0 T). \qquad (3.7)$$

and analogues

$$\| x'_{n+1}(.;\xi,\eta) - x'_n(.;\xi,\eta) \|_\infty \leq M \| f_n(s;\xi,\eta) - f_{n-1}(s;\xi,\eta) \|_1$$

$$\leq \frac{[M \| k \|_1]^n}{n!} (\| \alpha(\xi,\eta) \|_1 + \varepsilon_0 T). \qquad (3.8)$$

Since $(\xi, \eta) \to \alpha(\xi, \eta)$ is continuous it is locally bounded, and by (3.6) it follows that for every $(\xi, \eta) \in E_0 \times E$ the sequence $(f_n(.;\xi', \eta'))_{n\in N}$ satisfies the Cauchy condition uniformly with respect to (ξ', η') in some neighborhood of (ξ, η). Hence if we denote by $f(.;\xi, \eta)$ the limit of the sequence $(f_n(.;\xi, \eta))_{n\in N}$ then $(\xi, \eta) \to f(.;\xi, \eta)$ is continuous from $E_0 \times E$ into $L^1([0 ,T], E)$.

Analogously, by (3.7) and (3.8) it follows that $(x_n(.;\xi, \eta))_{n\in N}$ is a Cauchy sequence in $C^1([0 ,T], E)$ locally uniformly with respect to (ξ, η). Then denoting by $x(.;\xi, \eta)$ its limit, it follows that the map $(\xi, \eta) \to x(.;\xi, \eta)$ is continuous. Moreover, since$(x_n(.;\xi, \eta))_{n\in N}$ converge uniformly to $x(.;\xi, \eta)$ and since

$$d(f_n(t;\xi,\eta), F(t; x(t;\xi,\eta))) \leq k(t) \| x_n(t;\xi,\eta) - x(t;\xi,\eta) \|,$$

passing to limit along a subsequence of $(f_n(.;\xi, \eta))_{n\in N}$ converging poi wise to $f(.;\xi, \eta)$ we obtain that $f(t;\xi, \eta) \in F(t; x(t;\xi, \eta))$, for all $(\xi, \eta) \in E_0 \times E$, a.e. $t \in [0, T]$, since $F(., .)$ has closed values. Passing to the limit in (iv) we obtain

$$x(t;\xi,\eta) = C(t)\xi + S(t)\eta + \int_0^t C(t-s) f(t;\xi,\eta) ds$$

and the proof is completed.

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