TOPOLOGIES ON THE GRAPH OF THE EQUIVALENCE RELATION ASSOCIATED TO A GROUPOID

by
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Abstract. Let G be a topological groupoid r and d be the range map respectively the domain map of G. The relation u~v if there is x such that r(x) = u and d(x) = v is an equivalence relation on the unit space $G^{(0)}$. The graph of this equivalence relation can be regarded as a groupoid R, and can be endowed with different topologies. We shall prove that if the restriction of the range map to the isotropy group bundle of G is open then we can endow R with a locally compact topology such that the existence of a Haar system on G is equivalent to the existence of a Haar system on R.

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1. Introduction

Any groupoid G defines an equivalence relation on the unit space $G^{(0)}$. The graph R of this equivalence relation can be regarded as a groupoid called the principal groupoid associated with G If G is a topological groupoid then we can endow R with the product topology induced from $G^{(0)}\times G^{(0)}$. If the topology on G is locally compact then the product topology on R is locally compact if and only if the graph the equivalence relation is locally closed. On the other hand if we endow R with the product topology induced from $G^{(0)}\times G^{(0)}$ the existence of a Haar system on G does not necessarily imply the existence of a Haar system on R.

We shall endow R with the quotient topology from G as in [8] and we shall assume that the restriction the range map to the isotropy group bundle of G is open We shall prove that the quotient topology is locally compact and the existence of a Haar system on G is equivalent to the existence of a Haar system on R.

For establishing notation we include some definitions that can be found in several places (e.g. [7], [5]). A groupoid is a set G together with a distinguished subset $G^{(2)} \subset G\times G$ (called the set of compassable pairs), and two maps:

$$(x, y) \to xy \ [: G^{(2)} \to G] \ \text{(product map)}$$
$$x \to x^{-1} \ [: G \to G] \ \text{(inverse map)}$$

such that the following relations are satisfied:

1. If $(x, y) \in G^{(2)}$ and $(y, z) \in G^{(2)}$ then $(xy, z) \in G^{(2)}$, $(x, yz) \in G^{(2)}$ and $(xy)z = x(yz)$.

2. $(x^{-1})^{-1} = x$ for all $x \in G$.
3. For all $x \in G$, $(x, x^{-1}) \in G^{(2)}$, and if $(z, x) \in G^{(2)}$ then $(zx)x^{-1} = z$.
4. For all $x \in G$, $(x^{-1}, x) \in G^{(2)}$, and if $(x, y) \in G^{(2)}$ then $x^{-1}(xy) = y$.

The maps r and d on G, defined by the formulae $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$, are called the range and the source maps. It follows easily from the definition that they have a common image called the unit space of G which is denoted $G^{(0)}$. Its elements are units in the sense that $xd(x) = r(x)x = x$. It is useful to note that a pair (x, y) lies in $G^{(2)}$ precisely when $d(x) = r(y)$ and that the cancellation laws hold (e.g. $xy = xz$ if $y = z$).

The bres of the range and the source maps are denoted $G^u = r^{-1}(\{u\})$ and $G_v = d^{-1}(\{v\})$, respectively. Also for $u, v \in G^{(0)}$, $G_v^u = G^u \cap G_v$. More generally, given the subsets $A, B \subset G^{(0)}$, we define $G^A = r^{-1}(A)$, $G_B = d^{-1}(B)$ and $G_B^A = r^{-1}(A) \cap d^{-1}(B)$. G_A^A becomes a groupoid (called the reduction of G to A) with the unit space A, if we define $(G_A^A)^{(2)} = G^{(2)} \cap (G_A^A \times G_A^A)$.

For each unit u, $G_u^u = \{ x \mid r(x) = d(x) = u \}$ is a group, called isotropy group at u. The group bundle

$$\{ x \in G \mid r(x) = d(x) \}$$

is denoted G' and is called the isotropy group bundle of G.

If A and B are subsets of G, one may form the following subsets of G:

$$A^{-1} = \{ x \in G \mid x^{-1} \in A \}$$
$$AB = \{ xy \mid (x, y) \in G^{(2)} \cap (A \times B)\}$$

The relation u~v if $G_v^u \neq \Phi$ is an equivalence relation on $G^{(0)}$. Its equivalence classes are called orbits and the orbit of a unit u is denoted [u]. The quotient space for this equivalence relation is called the orbit space of G and denoted $G^{(0)}/G$. The graph of this equivalence relation will be denoted in this paper by

$$R = \{ (r(x), d(x)), x \in G \}$$

A groupoid is said transitive if and only if it has a single orbit or equivalently if the map $\theta : G \to G^{(0)} \times G^{(0)}$, $\theta(x) = (r(x), d(x))$ is surjective. A groupoid is said principal if the map θ is injective.

A topological groupoid consists of a groupoid G and a topology compatible with the groupoid structure. This means that:

1. $x \to x^{-1}$ [: $G \to G$] is continuous.
2. (x, y) [: $G \to G$] is continuous where $G^{(2)}$ has the induced topology from $G \times G$.

We are concerned with topological groupoids which are second countable, locally compact Hausdorff. It was shown in [6] that measured groupoids (in the sense of Definition 2.3./p. 6 [3]) may be assume to have locally compact topologies, with no loss in generality.

If X is a locally compact space, $C_c(X)$ denotes the space of complex-valuated continuous functions with compact support. The Borel sets of a topological space are taken to be the σ-algebra generated by the open sets.

We end this introductory section with a list of structures which fit naturally intothe study of groupoids:

1. *Groups*. A group G is a groupoid with $G^{(2)} = G \times G$ and $G^{(0)} = \{e\}$ (the unit element).
2. *Spaces*. A space X is a groupoid letting

$$X^{(2)} = \mathrm{diag}(X) = \{(x, x) , x \in G \}$$

and defining the operations by xx = x, and $x^{-1} = x$.

3. *Transformation groups.* Let Γ be a group acting on a set X such that for x $\in$ X and g $\in \Gamma$, xg denotes the transform of x by g. Let G = X $\times$ Γ, $G^{(2)} = \{ ((x, g) , (y, h)) : y = xg \}$. With the product (x, g) (xg, h) = (x, gh) and the inverse $(x, g)^{-1} = (xg, g^{-1})$ G becomes a groupoid. The unit space of G may be identified with X.

4. *Equivalence relations.* Let $E \subset X \times X$ be an equivalence relation on the set X. Let $E^{(2)} = \{((x_1, y_1), (x_2, y_2)) \in E \times E \mid y_1 = x_2\}$. With product (x, y) (y, z) = (x, z) and $(x, y)^{-1} = (y, x)$, E is a principal groupoid. $E^{(0)}$ may be identified with X. Two extreme cases deserve to be single out. If E = X $\times$ X then E is called the trivial groupoid on X, while if E = diag(X), diag X then E is called the co-trivial groupoid on X (and may be identified with the groupoid in example 2).

If G is any groupoid, then

$$R = \{(r(x), d(x)) \mid x \in G\}$$

is an equivalence relation on $G^{(0)}$. The groupoid defined by this equivalence relation is called the principal groupoid associated with G.

Any locally compact principal groupoid can be viewed as an equivalence relation on a locally compact space X having its graph $E \subset X \times X$ endowed with a locally compact topology compatible with the groupoid structure. This topology can be finer than the product topology induced from X $\times$ X. We shall endow the principal groupoid associated with a groupoid G with the quotient topology induced from G by the map

$$\theta : G \rightarrow R,\ \theta(x) = (r(x), d(x))$$

This topology consists of the sets whose inverse images by θ in G are open.

2. Continuous systems of measures

Let G be a locally compact groupoid, and R be the principal groupoid associated with G.

Throughout this section we shall fix a system of measures indexed on R.

$$\{ \beta_u^v , (u, v) \in R \}$$

satisfying the following conditions.

1. supp(β_u^v) = G_u^v for all u~v.
2. $\sup_{u,v} \beta_u^v(K) < \infty$ for all compact K ⊂ G.
3. $\int f(y)d\beta_v^{r(x)}(y) = \int f(xy)d\beta_v^{d(x)}(y)$ for all x ∈ G and v ~ r(x).

Different ways to constructing such systems of measure can be found in [1] (for transitive groupoids), [8], [2] (for locally compact second countable groupoids). The construction from [8] is sketched at the beginning of the next section.

Proposition 1. Let G be a locally compact second countable groupoid. Let us suppose that the map r': G' → $G^{(0)}$, r'(x) = r(x) is open.

Then for each f ∈ $C_c(G)$, the function

$$x \to \int f(y)d\beta_{d(x)}^{r(x)}(y)$$

is continuous on G.

Proof. By Lemma 1.3/p. 6 [8], for each f : G → **C** continuous with compact support the function u→ $\int f(y)d\beta_u^u(y)$ [: $G^{(0)}$ → **C**] is continuous. Let x ∈ G and $(x_i)_i$ be a sequence in G converging to x. Let f be a continuous function with compact support on G, and let g be a continuous extension on G of y → f(xy) [: $G^{d(x)}$ → **C**]. Let K be the compact set

$$(\mathrm{supp}(f)\ \{x, x_i, i = 1, 2, \ldots\}^{-1} \cup \mathrm{supp}(g)) \cap r^{-1}(\{d(x), d(x_i), i = 1, 2, \ldots\}).$$

We have

$$\left|\int f(y)d\beta_{d(x)}^{r(x)}(y) - \int f(y)d\beta_{d(x_i)}^{r(x_i)}(y)\right|$$

$$= \left|\int f(xy)d\beta_{d(x)}^{d(x)}(y) - \int f(x_i y)d\beta_{d(x_i)}^{d(x_i)}(y)\right|$$

$$= \left|\int g(y)d\beta_{d(x)}^{d(x)}(y) - \int f(x_i y)d\beta_{d(x_i)}^{d(x_i)}(y)\right|$$

$$\le \left|\int g(y)d\beta_{d(x)}^{d(x)}(y) - \int g(y)d\beta_{d(x_i)}^{d(x_i)}(y)\right| +$$

$$+ \left|\int g(y)d\beta_{d(x_i)}^{d(x_i)}(y) - \int f(x_i y)d\beta_{d(x_i)}^{d(x_i)}(y)\right|$$

$$\le \left|\int g(y)d\beta_{d(x)}^{d(x)}(y) - \int g(y)d\beta_{d(x_i)}^{d(x_i)}(y)\right| + \sup|g(y) - f(x_i y)|\ \beta_{d(x_i)}^{d(x_i)}(K)$$

A compactness argument shows that sup|g(y) − f(x_iy)| converges to 0. On the other hand, $\left|\int g(y)d\beta_{d(x)}^{d(x)}(y) - \int g(y)d\beta_{d(x_i)}^{d(x_i)}(y)\right|$ converges to 0, because the function u→ $\int f(y)d\beta_u^u(y)$ is continuous on $G^{(0)}$ and d : G → $G^{(0)}$ is also continuous. Hence

$$\left|\int f(y)d\beta_{d(x)}^{r(x)}(y) - \int f(y)d\beta_{d(x_i)}^{r(x_i)}(y)\right|$$

converges to 0.

Remark 2. If G is a locally compact second countable groupoid and if the map r' G' → $G^{(0)}$, r'(x) = r(x) is open, then for each f ∈ $C_c(G)$ the function

$$(u, v) \rightarrow \int f(y)\beta_v^u(y)$$

is continuous on R, where R is endowed with the quotient topology. Indeed from the Proposition 1 it follows that the composition of this map with θ is continuous on G.

3. A locally compact topology on the graph of the equivalence relation associated to a locally compact groupoid

Let G be a locally compact groupoid, and R be the principal groupoid associated with G. In Section 1 of [8] Jean Renault constructs a Borel Haar system for G'. One way to do this is to choose a function F_0 continuous with conditionally support which is nonnegative and equal to 1 at each u ∈ G_0. Then for each u ∈ $G^{(0)}$ choose a left Haar measure β_u^u on G_u^u so the integral of F_0 with respect to β_u^u is 1.

Renault defines β_v^u = xβ_v^v u if x ∈ G_v^u (where xβ_v^v(f) = $\int f(xy)\beta_v^v(y)$ as usual). If z is another element in G_v^u then $x^{-1}z$ ∈ G_v^v, and since β_v^v is a left Haar measure on G_v^v, it follows that β_v^u is independent of the choice of x. If K is a compact subset of G, then $\sup_{u,v} \beta_u^v(K) < \infty$. This system satisfies the following conditions:

1. supp(β_v^u) = G_v^u for all u~v.
2. $\sup_{u,v} \beta_v^u(K) < \infty$ for all compact K ⊂ G.
3. $\int f(y)d\beta_v^{r(x)}(y) = \int f(xy)d\beta_v^{d(x)}(y)$ for all x ∈ G and v ~ r(x).

Assuming that the map r' : G' → $G^{(0)}$, r'(x) = r(x) is open, and using the continuity of the map

$$(u, v) \to \int f(y)d\beta_v^u(y)$$

we shall prove that the map $\theta : G \to R$, $\theta(x) = (r(x), d(x))$ is open, and consequently the topology of R is locally compact. Using the same hypothesis, we shall also prove that the existence of a Haar system on G is equivalent to the existence of a Haar system on R.

Proposition 3. If G is a locally compact second countable groupoid and if the map r' G' $\to G^{(0)}$, r'(x) = r(x) is open, then the map

$$\theta : G \to R,\ \theta(x) = (r(x), d(x))$$

is open, where R is endowed with the quotient topology.

Proof . We have noted in Remark 2 that the map

$$(u, v) \to \int f(y)d\beta_v^u(y)$$

is continuous for every $f \in C_c(G)$.

Let D be an open set in G, let $x_0 \in D$ and $(u_0, v_0) = \theta(x_0)$. We can choose a nonnegative continuous function with compact support, f, that is equal to 1 on a compact neighborhood of x_0, and that vanishes outside D. The continuity of the map

$$(u, v) \to \int f(y)d\beta_v^u(y)$$

implies that the set of $(u, v) \in \theta(G)$ with $\int f(y)d\beta_v^u(y) \neq 0$ is open. This set is a neighborhood of (u_0, v_0) contained in $\theta(D)$.

Remark. If G is a locally compact second countable groupoid and if the map r' G' $\to G^{(0)}$, r'(x) = r(x) is open, then R is locally compact groupoid (with the quotient topology from G).

We have proved that the openness of r' implies the openness of θ. We shall prove that these conditions are in fact equivalent. We need the following lemma from [4], p.7.

Lemma 5. Let $f : X \to Z$ and $g : Y \to Z$ be two functions, and let

$$X*Y = \{ (x, y) \in X \times Y \mid f(x) = g(y) \}$$

Let π_X and π_Y be the projections of X*Y onto X and Y, respectively. The following diagram is commutative

$$\begin{array}{ccc} X*Y & \xrightarrow{\pi_X} & X \\ \pi_Y \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

and $\pi_Y((U \times V) \cap (X^*Y)) = g^{-1}(f(U)) \cap V$ for all $U \subset X$ and $V \subset Y$.

Proposition 6. Let G be a locally compact groupoid. If θ is open for the quotient topology, then the map $\delta' : G *_r G' \to G$, $\delta'(x, y) = x^{-1}y$ is open where

$$G *_r G' = \{ (x, y) \in G \times G' \mid r(x) = r(y) \}$$

is endowed with the topology induced from $G \times G$.

Proof. Since the inverse map $x \to x^{-1}$ is a homeomorphism from G to G and θ is an open map it follows that the map $x \to \theta(x^{-1})$ is an open map from G to R endowed with the quotient topology. Let

$$G^*G = \{ (x, y) \in G \times G \mid \theta(x) = \theta(y^{-1}) \}$$

It is not hard to see that

$$(x, y) \to (x, x^{-1}y) \ [: G^*G' \to G^*G]$$

is a homeomorphism. Therefore for proving that δ' is open it is enough to show that

$$(x, y) \xrightarrow{\pi_2} y \ [: G^*G \to G]$$

is open. We consider the following commutative diagram:

$$\begin{array}{ccc} G^*G & \xrightarrow{\pi_1} & G \\ \pi_2 \downarrow & & \downarrow \theta \\ G & \xrightarrow{y \to \theta(y^{-1})} & R \end{array}$$

Let U and V two open subsets of G. Applying Lemma 5 and Proposition 3 we obtain that

$$\pi_2((U^*V) \cap (G \times G')) = \theta^{-1}(\theta(U)^{-1}) \cap V = \theta^{-1}(\theta(U^{-1})) \cap V,$$

is open. Since the sets of the form $(U \times V) \cap (G^*G)$ constitute a basis for the topology on G^*G, it follows that π_2 is open.

Corollary 7. Let G be a locally compact groupoid and R be the associated principal groupoid endowed with the quotient topology. If the map $\theta : G \to R$, $\theta(x) = (r(x), d(x))$ is open then for each open subset U of G, UG' is open in G.

Proposition 8. Let G be a locally compact groupoid and R be the associated principal groupoid endowed with the quotient topology. If the map $\theta : G \to R$, $\theta(x) = (r(x), d(x))$ is open then the map r' $G' \to G^{(0)}$, $r'(x) = r(x)$ is open.

Proof. Applying the preceding corollary we obtain that UG' is open in G, for each open subset U of G. Consequently, $G'U^{-1} = (UG')^{-1}$ is open in G. Now the openness of the map r' follows noticing that

$$r'(U \cap G') = G'U \cap G^{(0)},\ U \subset G.$$

Proposition 9. Let G be a locally compact second countable groupoid and R be the associated principal groupoid endowed with the quotient topology. Then the map r' G' $\to G^{(0)}$, r'(x) = r(x) is open if and only if $\theta : G \to R$, $\theta(x) = (r(x), d(x))$ is open.

Proof. It follows from Remark 2 and Proposition 8.

Remark 10. If G is a principal topological groupoid then the map r' G' $\to G^{(0)}$, r'(x) = r(x) is open. Indeed, to see this, we just notice that

$$r'(U \cap G') = U \cap G^{(0)}$$

for any subset U of G.

Definition 11. A Haar system on a locally compact groupoid G is a family of positive Radon measures on G, $\{\nu^u \in G^{(0)}\}$, having the following properties:

1. For all $u \in G^{(0)}$ $\mathrm{supp}(\nu^u) = G^u$.
2. For all $f \in C_c(G)$
   $$u \to \int f(x) d\nu^u(x) \quad [: G^{(0)} \to \mathbf{C}]$$
   is continuous.
3. For all $f \in C_c(G)$ and all $x \in G$,
   $$\int f(y) d\nu^{r(x)}(y) = \int f(xy) d\nu^{d(x)}(y).$$

The system of measures $\{\nu^u \in G^{(0)}\}$ will be called Borel Haar system if it has the properties 1., 3. and

2' For all $f \geq 0$ Borel on G,

$$u \to \int f(x) d\nu^u(x) \quad [: G^{(0)} \to \overline{R}]$$

is a real extended Borel map, where the Borel sets of a topological spaces G and $G^{(0)}$ are taken to be the algebra generated by the open sets.

Proposition 12. Let G be a second countable locally compact groupoid which admits a Haar system $\{\nu^u \in G^{(0)}\}$. Let R be the associated principal groupoid endowed with the quotient topology. If the map r' G' $\to G^{(0)}$, r'(x) = r(x) is open, then R admits a Haar system.

Proof . By Lemma 1.7/p. 9 [8] there is a unique Borel Haar system α on R with the property that for every u $\in$ $G^{(0)}$ we have:

$$\nu^u = \int \beta_v^u d\alpha^u(w,v).$$

Let g be a function on R continuous with compact support for the quotient topology. Since G is locally compact and θ is open from G to R, there is a compact subset K of G such that $\theta(K)$ contains the support of h. Let $F_1 \in C_c(G)$ be a nonnegative function equal to 1 on a compact neighborhood U of K. Let $F_2 \in C_c(G)$ be a function which extends to G the function x $\to$ $F_1(x) / \int F_1(y)d\beta_{d(x)}^{r(x)}(y)$, x $\in$ U. We have $\int F_2(y)d\beta_v^u(y) = 1$ for all (u, v) $\in$ $\theta(K)$. Since

$$\int g(w,v)d\alpha^u(w,v) = \int g(w,v) \int F_2(y)d\beta_v^u(y)d\alpha^u(w,v)$$

$$= \int g(r(y),d(y))F_2(y)d\nu^u(y),$$

it follows that u $\to$ $\int g(w,v)d\alpha^u(w,v)$ is continuous.

Proposition 13. Let G be a second countable locally compact groupoid for which the map r' G' $\to$ $G^{(0)}$, r'(x) = r(x) is open. Let R be the associated principal groupoid endowed with the quotient topology. If R admits a Haar system { α^u $\in$ $G^{(0)}$ } then G admits a Haar system.

Proof. If we define

$$\nu^u = \int \beta_v^w d\alpha^u(w,v)$$

then { ν^u $\in$ $G^{(0)}$ } is Haar system for G.

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