ON SOME ASPECTS REGARDING THE GENERALIZATION OF THE OPTIMAL FORMULAS OF SARD TYPE USING THE BOOLEAN-SUM TYPE OPERATORS

by
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Abstract. In this paper will be presented some results about the optimal quadrature's formulas of Sard type and there, starting from considerations, a generalization for the cubature's formulas of boolean-sum type builts with the help of some operators who satisfy the optimal formulas mentioned above. A better generalization can be obtained using a boolean-sum type formula with a number of n operators and n corresponding remainders.

We shall consider the next quadrature' formula:

$$\int_a^b f(x)d\lambda(x) = \sum_{k=0}^{m} A_k f(a_k) + R_m(f) \qquad (1),$$

where $d\lambda$ is a positive measure of the integration integral , a_k are the nodes of the quadrature's formula and satisfy the relations:

$$a \le a_0 < a_1 < ... < a_m \le b,$$

A_k are named the coefficients of the formula
and $R_m(f)$ are named the remainder

Theorem 1

If $f \in H^n[a,b]$ if we use the Peano's theorem we shall have the next representation of the remainder:

$$R_m(f) = \int_a^b K_{m,n}(t) f^{(n)}(t)dt \qquad (2),$$

where

$$K_{m,n}(t) = R_m\left[\frac{(x-t)_+^{n-1}}{(n-1)!}\right] = \frac{1}{(n-1)!}\cdot\left[\int_a^b (x-t)_+^{n-1} d\lambda(x) - \sum_{k=0}^{m} A_k (a_k - t)_+^{n-1}\right]$$

is named the Peano's kernel.

Remark 1

If the Peano's kernel has a constant sign on the entire interval of the integration then using a midle formula we shall can write:

$$R_m[f] = f^{(m)}(\xi)\int_a^b K_{m,n}(t)dt, \quad a < \xi < b. \tag{3}$$

Definition 1

The quadrature's formula given by (1) will be named optimal of Sard type if:

$$\int_a^b \left|K_{m,n}(t)\right|^2 dt \to \min \tag{4}$$

Theorem 2

If $n \le m$ the quadrature's formula given by (1) who is considerated optimal of Sard type has an one solution and this solution will be obtained by the integration of the spline interpolation formula

$$f(x) = (Sf)(x) + R_m(f,x), x \in [a,b]$$

that means

$$\int_a^b f(x)d\lambda(x) = \int_a^b (Sf)(x)d\lambda(x) + \int_a^b R_m(f,x)d\lambda(x) \quad (5).$$

We shall consider now some questions about the generalization of the optimal quadrature's formulas of Sard type, more precisely about the cubature formulas of boolean-sum type who results from optimals quadrature's formulas of Sard type.

Let's consider the next decomposition of the unit operator:

$$\begin{aligned} I &= P_1 \oplus P_2 + R_1R_2 = \\ &= (P_1 + P_2 - P_1P_2) + R_1R_2 \end{aligned} \tag{6}$$

where P_1, P_2 are interpolation polynoms and R_1, R_2 are the corresponding remainders:

$$I = P_1 + R_1, I = P_2 + R_2 \quad (7).$$

Remark 2

If we shall integrate the formula given by (6) on a plane domain $D=[a,b]x[c,d]$, where P_1, P_2 are supposed to be defined on $[a,b]$, respectively $[c,d]$, we will obtain a cubature formula with the remainder given by:

$$\iint_{D=[a,b]x[c,d]} (R_1 R_2 f)(x,y)dxdy$$

On R_1, R_2 we shall suppose that they have the corresponding Peano's kernels K_{m_1,n_1}, K_{m_2,n_2} given by:

$$K_{m_1,n_1}(t)=R_{m_1}\left[\frac{(x-t)_+^{n_1-1}}{(n_1-1)!}\right]=\frac{1}{(n_{1^{\cdot}}-1)!}\cdot\left[\int_{a_1}^{b_1}(x-t)_+^{n_1-1}d\lambda(x)-\sum_{k=0}^{m_1}A_{k1}(a_{k_1}-t)_+^{n_1-1}\right],$$

$$K_{m_2,n_2}(t)=R_{m_2}\left[\frac{(x-t)_+^{n_2-1}}{(n_2-1)!}\right]=\frac{1}{(n_2-1)!}\cdot\left[\int_{a_2}^{b_2}(x-t)_+^{n_2-1}d\lambda(x)-\sum_{k_2=0}^{m_2}A_{k_2}(a_{k_2}-t)_+^{n_2-1}\right]$$

Definition 2

About a cubature's formula generated by the integration of a boolean-sum type formula we shall say that is optimal of Sard type if it verifies the relation:

$$\iint_D \left|K_{m_1,n_1}(s)\right|^2 \left|K_{m_2,n_2}(t)\right|^2 dsdt \to \min$$

Theorem 3

If P_1, P_2 are the operators corresponding to an optimal quadrature's formula and R_1, R_2 the corresponding remainders then the cubature's formula obtained by the integration of a (6) type formula is optimal of Sard type in the sens of the definition 2.

Remark 3

In order to demonstrate theorem 3 first is easily to see that

$$\iint_D \left|K_{m_1,n_1}(s)\right|^2 \left|K_{m_2,n_2}(t)\right|^2 dsdt = \int_a^b \left|K_{m_1,n_1}(s)\right|^2 ds \cdot \int_c^d \left|K_{m_2,n_2}(t)\right|^2 dt \qquad (8)$$

If we take care that the quadrature's formulas generated from the P_1, P_2 operators and the corresponding remainders R_1, R_2 are optimals of Sard type, that means

$$\int_a^b \left|K_{m_1,n_1}(s)\right|^2 ds \to \min$$
$$\int_c^d \left|K_{m_2,n_2}(t)\right|^2 dt \to \min$$

will be evident taking care of (8) that

$$\iint_D \left|K_{m_1,n_1}(s)\right|^2 \left|K_{m_2,n_2}(t)\right|^2 dsdt \to \min$$

Remark 4

Such types of optimals quadrature's formulas can be generalizated starting from a number of n operators $P_1, P_2, ..., P_s$ and the corresponding remainders $R_1, R_2, ..., R_s$ using an n-dimensional boolean-sum type formula:

$$I = P_1 \oplus P_2 \oplus ... \oplus P_s + R_1 R_2 ... R_s . \qquad (9)$$

Remark 5

A such type of boolean-sum type formula can be obtained supposing that we have verifieds the conditions

$$P_i P_j = P_j P_i, \forall i, j = 1, ..., n, i \neq j$$

and using a recursive formula

$$P_1 \oplus P_2 \oplus ... \oplus P_n = \left(P_1 \oplus P_2 \oplus ... \oplus P_{n-1}\right) \oplus P_n . \qquad (10)$$

For example, using a number of 3 operators P_1, P_2, P_3 we shall have:

$$P_1 \oplus P_2 \oplus P_3 = \left(P_1 \oplus P_2\right) \oplus P_3 = P_1 + P_2 + P_3 - P_1 P_2 - P_1 P_3 - P_2 P_3 + P_1 P_2 P_3 .$$

Remark 6

It is easily to verify that the formula given by (10) has the property of associativity.

Definition 3

About a cubature's formula generated by the integration of a boolean-sum type formula given by (9) we shall say that is optimal of Sard type if it verifies the relation:

$$\int\int\ldots\int_D \left|K_{m_1,n_1}(t_1)\right|^2 \left|K_{m_2,n_2}(t_2)\right|^2 \ldots \left|K_{m_s,n_s}(t_s)\right|^2 dt_1 dt_2 \ldots dt_s \to \min$$

with $D = [a_1, b_1]x[a_2, b_2]x\ldots x[a_s, b_s]$.

Supposing that the s quadrature's formulas builts with the operators and the remainders mentionated above are optimal of Sard type, that means:

$$\begin{aligned}
&\int_{a_1}^{b_1} \left|K_{m_1,n_1}(t)\right|^2 dt \to \min\\
&\int_{a_2}^{b_2} \left|K_{m_2,n_2}(t)\right|^2 dt \to \min \qquad (11)\\
&\ldots\ldots\\
&\int_{a_s}^{b_s} \left|K_{m_s,n_s}(t)\right|^2 dt \to \min
\end{aligned}$$

where

$$K_{m_1,n_1}(t) = R_{m_1}\left[\frac{(x-t)_+^{n_1-1}}{(n_1-1)!}\right] = \frac{1}{(n_1-1)!}\cdot\left[\int_{a_1}^{b_1}(x-t)_+^{n_1-1}d\lambda(x) - \sum_{k=0}^{m_1} A_{k1}\left(a_{k_1}-t\right)_+^{n_1-1}\right]$$

$$K_{m_2,n_2}(t) = R_{m_2}\left[\frac{(x-t)_+^{n_2-1}}{(n_2-1)!}\right] = \frac{1}{(n_2-1)!}\cdot\left[\int_{a_2}^{b_2}(x-t)_+^{n_2-1}d\lambda(x) - \sum_{k_2=0}^{m_2} A_{k_2}\left(a_{k_2}-t\right)_+^{n_2-1}\right]$$

$$\ldots$$

$$K_{m_s,n_s}(t) = R_{m_s}\left[\frac{(x-t)_+^{n_s-1}}{(n_s-1)!}\right] = \frac{1}{(n_s-1)!}\cdot\left[\int_{a_s}^{b_s}(x-t)_+^{n_s-1}d\lambda(x) - \sum_{k_s=0}^{m_s} A_{k_s}\left(a_{k_s}-t\right)_+^{n_s-1}\right]$$

are the corresponding Peano kernels it is easy to see that we have

$$\int_D \left|K_{m_1,n_1}(t_1)\right|^2 \ldots \left|K_{m_s,n_s}(t_s)\right|^2 dt_1 \ldots dt_s = \int_{a_1}^{b_1} \left|K_{m_1,n_1}(t_1)\right|^2 dt_1 \ldots \int_{a_s}^{b_s} \left|K_{m_s,n_s}(t_s)\right|^2 dt_s \to \min$$

where $D = [a_1, b_1]x[a_2, b_2]x \ldots x[a_s, b_s]$

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