# FORMS OF THE CAYLEY-HAMILTON THEOREM FOR GENERALIZED SYSTEMS

by
**Basil G. Mertzios**

**Abstract**: The Cayley_Hamilton Theorem for generalized systems is derived for the constant square matrices $A_0, A_1, \dots, A_K$ ; $k \geq 1$, by applying the fundamental idea of the CH Theorem for the polynomial matrix $A(s) = s^k A_K + \dots + sA_1 + A_0 \in R[s]^{nxn}$. **A**nother natural extension of the Cayley-Hamilton Theorem to Generalized (or Singular, or Semistate, or Descriptor) and Matrix Fraction Description (MFDs) Systems is based on the use of the fundamental matrix sequence $\mathbf{\Phi_i}$, which are the matrix coefficients of the Laurent expansion of the Laplace transform $\mathbf{\Phi}(s)$ of the fundamental matrix $\mathbf{\Phi}(t)$. The fundamental matrix coefficient matrices $\mathbf{\Phi_i}$ of the generalized system having system's matrices $A_i$ i=0,1,…k, $k \geq 1$ may be analytically and recursively computed in terms of system's matrices. The *Copyright © 2003 IFAC.*
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## 1. INTRODUCTION

The Cayley Hamilton (CH) Theorem is valid for matrices over a commutative ring R and it says that a matrix satisfies the characteristic equation.

In the context of system theory, the CH Theorem in for regular state-space systems is expressed in terms of the powers of the system's matrix. $\mathbf{A}^i$, i=0,1,…,n. Moreover, since, $\mathbf{A^i} = \mathbf{\Phi_i}$, where $\mathbf{\Phi_i}$ are the coefficients of the Laurent expansion at infinity of the Laplace transform $\mathbf{\Phi}(s)=(s\mathbf{I} - \mathbf{A})^{-1}$ of the fundamental matrix $\mathbf{\Phi}(t)$, the CH Theorem may be seen as a linear relation among $(n+1)$ consecutive fundamental matrix coefficient matrices $\mathbf{\Phi}_i$.

In this paper the CH Theorem is presented for the case of two or more square matrices $\mathbf{A}_i$; i=0,1,…k, $k \geq \mathbf{1}$, or equivalently for the generalized systems, in which two or more square system matrices are involved. Two distinct forms based on the algebraic and the systems' theory approaches are considered. The underlying relation of both approaches, as well as their advantages is discussed.

The generalized CH theorem is useful for the calculation of the state transition matrix of the system and therefore for the solution of the associated homogeneous matrix differential $\mathbf{A}(D)\, \mathbf{x}(t)=\mathbf{0}$, equation, for the calculation of the controllability and observability Grammians of the system (Kailath, 1980), as well as in the analysis and synthesis procedures of the generalized systems.

The algebraic approach for the derivation of the CH Theorem is based on the characteristic equation of the polynomial matrix $A(s)=s^k\,\mathbf{A}_k+\dots+s\mathbf{A}_1+\mathbf{A}_0 \in R[s]^{nxn}$ in one variable s, having as coefficients the real square matrices $\mathbf{A}_i$; i=0,1,…k, k ≥ **1** and provides direct relations among the powers of the matrices. The CH Theorem for two or more square matrices may be reduced to the CH Theorem of the polynomial ring

On the other hand, in the system theory framework, the CH Theorem for generalized systems is expected to be a linear relation among consecutive fundamental matrix coefficient matrices $\mathbf{\Phi}_i$ of the generalized systems, having system matrices $\mathbf{A}_i$; i=0,1,…k, k ≥ 1 Therefore, a natural extension of the C-H Theorem in Singular and Matrix Fraction Description Systems (MFDs) is to use the fundamental matrix sequence $\mathbf{\Phi}_i$, which involve $\mathbf{A}_i$; i=0,1,…k, k ≥ 1. Since $\mathbf{\Phi}_i$ may be analytically and recursively computed in terms of $\mathbf{A}_i$; i=0,1,…k, k ≥ 1, the CH direct relation may be also derived.

The generalized or singular systems consist of both dynamical differential and algebraic equations and are characterized by impulsive behavior. Such systems arise in the study of inversion of state-space systems (Silverman, 1969), in large scale interconnected systems (Rosenbrock and Pugh, 1974; Singh and Liu, 1973), in using proportional-plus-derivative control laws in state-space systems (Armentano, 1985), in power systems (Stott, 1979), in economics (Leontieff model) (Luenberger and Arbel, 1977), in demography models (Leslie model) (Campbell 1980, 1982), in network theory (Newcomb, 1981), in biology, e.t.c.

The first attempt to formulate the CH Theorem for two constant square matrices, **E,A,** based on the algebraic approach, has been presented in (Mertzios and Christodoulou, 1986). Moreover, the CH Theorem may be expressed in terms of the fundamental matrix sequence $\mathbf{\Phi}_i$ of the associated generalized singular system (Lewis, 1986). In singular systems $\mathbf{\Phi}(s)$ is a generalized matrix pencil $\mathbf{\Phi}(s) = (s\mathbf{E}-\mathbf{A}) \in \mathbf{R}[s]^{nxn}$. Then the fundamental sequence $\mathbf{\Phi}_i$ satisfies the characteristic equation of the associated pencil. The explicit and recursive calculation of $\mathbf{\Phi}_i$ in terms of the generalized system's matrices **E,A,** has been addressed in (Mertzios and Lewis, 1989).

For completeness and in order to introduce properly the notation, the relation of the fundamental matrix and of the CH Theorem is reviewed for the regular state-space systems in Section 2. In Section 3 and 4 two forms of the CH theorem are extracted for the generalized systems and for the Matrix Differential Systems respectively. The first form is extracted by extending the algebraic based approach by considering the characteristic equation of a polynomial matrix, having as coefficients k ≥ 2 square matrices. The second form is extracted using the systems' approach and is expressed in terms of the fundamental matrix sequence, which depends on the k ≥ 2 system's matrices.

## 2. RELATION OF THE CAYLEY-HAMILTON THEOREM WITH THE FUNDAMENTAL MATRIX AND MARKOV PARAMETERS

In this section, the relation of the fundamental matrix sequence and of the CH Theorem is reviewed for the regular state-space systems of the form

$$\mathbf{X}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t),\ \mathbf{y}(t)=\mathbf{C}\mathbf{x}(t) \tag{1}$$

The resolvent matrix of (1) is the inverse matrix $\mathbf{A}^{-1}(s)=(s\mathbf{I}_n-\mathbf{A})^{-1}$, which is a strictly proper rational matrix expressed as (Kailath, 1980)

$$\mathbf{A}^{-1}(s)=(s\mathbf{I}_n-\mathbf{A})^{-1}=\frac{1}{p(s)}\mathbf{B}(s) \tag{2}$$

where

$$\mathbf{B}(s)=\mathrm{Adj}(s\mathbf{I}-\mathbf{A})=\in\sum_{m=0}^{(n-1)}\mathbf{B}_{n-1-m}S^m$$

$$=\mathbf{I}_n S^{n-1}+\mathbf{B}_{n-2}s+\ldots+\mathbf{B}_0 s^{n-1} \tag{3}$$

$$p(s)=\det(s\mathbf{I}_n-\mathbf{A})=\sum_{m=0}^{n}p_{n-m}S^m=s^n+p_{n-1}s+\ldots+p_0 s^n \tag{4}$$

Then the application of the classical Leverrier algorithm gives

$$p_m=-\frac{1}{m}\mathrm{tr}[\mathbf{A}\mathbf{R}_{m-1}];\ m=1,2,\ldots,n \tag{5a}$$

$$\mathbf{R}_m=\mathbf{A}\mathbf{R}_{m-1}+p_m\mathbf{I}_n;\ m=1,2,\ldots,n-1 \tag{5b}$$

with initial conditions $B_{n-1}=\mathbf{I}_n$, $p_n=1$. According to the CH Theorem, a single square matrix **A**, satisfies its characteristic equation

$$p(\mathbf{A})=\Delta[\mathbf{A}]=\mathbf{A}^{\mathbf{n}}+p_1\mathbf{A}^{n-1}+\ldots+p_{n-1}\mathbf{A}+\mathbf{p}_n\mathbf{I}_n=0 \tag{6}$$

Another forms of the C-H Theorem is given by the *n*th recursion of the form (5b), which gives

$$P(\mathbf{A})=\mathbf{R}_n=\mathbf{A}\mathbf{R}_{n-1}+\mathbf{p}_n\mathbf{I}_n=\mathbf{0} \tag{7}$$

and from the matrix equation of the form

$$p(\mathbf{A})=\mathbf{\Phi}_k+p_1\mathbf{\Phi}_{k-1}+\ldots+p_{n-1}\mathbf{\Phi}_{k-n+1}+p_n\mathbf{\Phi}_{k-n}=\mathbf{0}\quad \text{for } k\geq n \tag{8}$$

where, $\mathbf{\Phi}_i=\mathbf{A}^i$, i $\geq\mathbf{0}$, is the fundamental matrix sequence, defined by the Laurent expansion of the resolvent matrix $(s\mathbf{I}_n-\mathbf{A})^{-1}$ at infinity, as follows:

$$\mathbf{A}^{-1}(s)=(s\mathbf{I}-\mathbf{A})^{-1}=\sum_{i=0}^{\infty}\mathbf{A}^i s^{-i-1}=\sum_{i=0}^{\infty}\mathbf{\Phi}_i s^{-i-1} \qquad (9)$$

since for the regular state space systems, $\mathbf{A}^{-1}(s)$ is a strictly proper rational matrix functions it is seen from (2).

The alternative identical formulation (8) of the CH theorem reveals the fact that the CH Theorem in system theory represents a recursive relation among the fundamental matrix coefficients $\mathbf{\Phi}_i$.

The system's Markov parameters are given in terms of $\mathbf{\Phi}_i=\mathbf{A}^i$, as follows:

$$\mathbf{H}_i=\mathbf{C}\mathbf{\Phi}_i\mathbf{B}=\mathbf{C}\mathbf{A}^i\mathbf{B};\ i=0,1,2,\ldots \qquad (10)$$

and are sufficient for the system's characterization and realization. Also the CH theorem in the form of (8) provides a recursive relation for the Markov parameters. $\mathbf{H}_i=\mathbf{C}\mathbf{\Phi}_i\mathbf{B}$, i $\geq 0$.

## 3. THE CAYLEY-HAMILTON THEOREM IN SINGULAR SYSTEMS

Let

$$\mathbf{E}\mathbf{x}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t);\ \mathbf{E}\mathbf{x}(t_0^-)=\mathbf{E}\mathbf{x}_0 \qquad (11a)$$

$$\mathbf{Y}(t)=\mathbf{C}\mathbf{x}(t) \qquad (11b)$$

be the generalized state space of a singular system. The generalized order of system (1), i.e. the number of degrees of freedom of the system, or equivalently the number of independent values of $\mathbf{E}\mathbf{x}(t_0^-)$ equals. *rank* ($\mathbf{E}$) $\leq$n-1. The inverse $\mathbf{A}^{-1}(s)$ of the generalized pencil $\mathbf{A}(s)=(s\mathbf{E}-\mathbf{A})$ represents the resolvent matrix of system (11) and gives rise to the generalized transfer function matrix $\mathbf{H}(s)=\mathbf{C}(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}$

### *3.1 Algebraic approach (Characteristic Equation)*

The resolvent matrix of (11) is the inverse matrix $\mathbf{A}^{-1}(s)=(s\mathbf{E}-\mathbf{A})^{-1}$ which is a non proper rational matrix expressed as (Campbell, 1977; Cobb, 1981; Dai, 1989; Lewis, 1986; Verghese, 1981).

Consider the square pencil $\mathbf{A}(s)=(s\mathbf{E}-\mathbf{A})\in\mathbf{R}\,[s]^{nxn}$. Then the characteristic equation of $\mathbf{A}(s)$ is

$$c(s,z)=\det[z\mathbf{I}_n-\mathbf{A}(s)]=z^n+c_1(s)z^{n-1}+\ldots+c_{n-1}(s)z+c_n(s)=0 \qquad (12)$$

where $c_i(s)$; i=1,2,…,n, are the coefficient polynomials appearing in the two-variable characteristic polynomial c(s,z) of **A**(s).

Extending the underlying idea of the Cayley-Hamilton Theorem, $\mathbf{A}(s)\in\mathbf{R}[s]^{nxn}$ satisfies its characteristic equation (Mertzios et.al., 1986)

$$C[s,\mathbf{A}(s)]=\mathbf{A}^n(s)+c_1(s)\mathbf{A}^{n-1}(s)+\ldots+c_{n-1}(s)\mathbf{A}(s)+c_n(s)\mathbf{I}_n=0 \qquad (13)$$

for all s, which represents a polynomial matrix equal to zero. Therefore all its coefficient matrices of the powers of s are zero.

It is seen from (13) that

$$C(s,z)\Big|_{z=0}=c(s,0)=\det[-\mathbf{A}(s)]=c_n(s) \qquad (14)$$

So that $c_n(s)\mathbf{I}_n=\mathrm{Adj}(-\mathbf{A}(s))(-\mathbf{A}(s))$.

In view of the Faddeva (Leverrier) algorithm for regular state-systems (Gantmacher, 1959; Kailath, 1980) and substituting the constant matrix **A** with the pencil **A**(s)=(s**E**-**A**), we may write the following:

Algorithm 1: Initial conditions

$$\mathbf{T}_0(s)=\mathbf{I}_n,\ c_0(s)=1 \qquad (15)$$

Recursions

$$\mathbf{T}_i(s)=\mathbf{A}(s)\mathbf{T}_{i-1}(s)+c_i(s)\mathbf{I}_n,\ i=1,2,\ldots n-1 \qquad (16)$$

$$C_i(s)=-\frac{1}{i}\,\mathrm{tr}[\mathbf{A}(s)\mathbf{T}_{i-1}(s)],\ i=1,2,\ldots,n \qquad (17)$$

Terminate

$$\mathbf{T}_{i-1}(s)=\mathrm{Adj.}\,[-\mathbf{A}(s)]=\mathrm{Adj.}\,[-s\mathbf{E}+\mathbf{A}] \qquad (18)$$

Final Condition

$$\mathbf{T}_n(s)=\mathbf{A}(s)\mathbf{T}_{n-1}(s)+c_n(s)\mathbf{I}_n=\mathbf{A}^n(s)+c_1(s)\mathbf{A}^{n-1}(s)+\ldots+c_{n-1}(s)\mathbf{A}(s)+ +c_n(s)\mathbf{I}_n=c[s,\mathbf{A}(s)]=0 \qquad (19)$$

It is seen from (15)-(17) theat the degree of the polynomial matrices $\mathbf{T}_i(s)$, i=0,1,…,n and of the polynomials $c_i(s)$; i=1,2,…n is at most equal to i and may be written in the form

$$\mathbf{T}_i(s)=\mathbf{A}^i(s)+c_1(s)\mathbf{A}^{i-1}(s)+\ldots+c(s)\mathbf{I}_n=\sum_{m=0}^{i}\mathbf{T}_{j,i-m}s^m=\mathbf{T}_i s^i+\mathbf{T}_{i,1}s^{i-1}+\ldots+\mathbf{T}_{i,j}, \qquad (20)$$

$$c_i(s)=\sum_{m=0}^{i}c_{i,i-m}s^m=c_{i,0}s^i+c_{i,1}s^{i-1}+\ldots+c_{i,i},\quad i=1,2,\ldots,n \qquad (21)$$

Using (20) and (21), the characteristic polynomial and the Adjoint matrix of (-**A**(s)) = (-s-**E**+**A**) are written as

$$c_n(s)=\sum_{m=0}^{r} c_{n,n-m}s^m = c_{n,n-r}s^r + c_{n,n-r+1}s^{r-1}+...c_{n,m} \tag{22}$$

$$\mathbf{T}_{n-1}(s)\sum_{m=0}^{f}\mathbf{T}_{n-1,n-1-m}s^m = \mathbf{T}_{n-1,n-1-f}s^{f-1}+...+\mathbf{T}_{n-1,n-1} \tag{23}$$

where the integers f and r are defined by

$$f \overset{\Delta}{=} \text{degree Adj } [-\mathbf{A}(s)]=\text{degree Adj } [-s\mathbf{E}+\mathbf{A}]=\text{degree } \mathbf{T}(s)=\leq \text{rank } (\mathbf{E}) \leq \text{n-1} \tag{24}$$

$$r \overset{\Delta}{=} \text{degree det } [-s\mathbf{E}+\mathbf{A}]=\text{degree c(s)} \leq \text{f} \leq \text{rank } (\mathbf{E}) \leq \text{n-1} \tag{25}$$

In view of (19) and (20), relation (13) may be written in the form

$$\mathbf{T}_{n,0}(\mathbf{E}, \mathbf{A})s^n+\mathbf{T}_{n,1}(\mathbf{E},\mathbf{A})s+\ldots+\mathbf{T}_{n,n}(\mathbf{E}, \mathbf{A})=0 \tag{26}$$

where $\mathbf{T}_{n,j}(\mathbf{E}, \mathbf{A})$, j=0,1,…,n are matrix functions of **E, A.**

Moreover, the power of the polynomial matrix $\mathbf{A}(s) \in \mathbf{R}[s]^{nxn}$ may be written as follows

$$\mathbf{A}^i(s)=(s\mathbf{E}-\mathbf{A})^i=\sum_{m=0}^{i}\mathbf{A}_{i,m}s^m, i=\mathbf{0},1,...,n \tag{27}$$

with the initial conditions

$$\mathbf{A}^0(s)=\mathbf{A}_{0,0}=\mathbf{I}_n, \mathbf{A}_{1,m}=\mathbf{A}_m, m=0,1 \tag{28}$$

The substitution of (21) and (27) in (13) results to matrix convolution equations, which are the matrix equalities expressed in terms of the coefficient matrices $\mathbf{A}_{i,m} \in \Re^{nxn}; i=0,1,...,n; m=0,1,...,i$, and are defined

$$\mathbf{T}_{n,j}=c_{n,j}\mathbf{I}+\sum_{v=0}^{n-1}\sum_{u=\max\{v-j,0\}}^{\min\{v,n-j\}}c_{v,v-u}\mathbf{A}_{n-v,n-j-u}=0, \quad j=0,1,\ldots nk \tag{29}$$

where is $\mathbf{A}_{i,m} \in \Re^{nxn}$; i=2,3,…,n; m=1,2,…,i is calculated in terms of in terms of **E**,**A**, by

$$\mathbf{A}_{i,m}=(-1)^{i-m} < \mathbf{E}^m, \mathbf{A}^{i-m} > \tag{30}$$

where the symbol $<\mathbf{E}^{\mu}, \mathbf{A}^{v}>$ denotes the sum of all $\begin{pmatrix} \mu+v \\ \mu \end{pmatrix} = [(\mu+v)!]/(\mu! v!)$ therms (the number of combinations of (μ+ v) times being taken μ each time), which consist of the product of matrices **E** and **A** appearing μ and v times respectively, i..e.

$$<\mathbf{E}^{\mu}, \mathbf{A}^{v}> = \mathbf{E}^{\mu}\mathbf{A}^{v} + \mathbf{E}^{\mu-1}\mathbf{A}^{v}\mathbf{E} + \ldots + \mathbf{A}^{v}\mathbf{E}^{\mu} \tag{31}$$

At this point the inversion algorithm of the generalized pencil **A**(s)=(s**E**-**A**) is presented, which is based on using (20) and (21) in the 1-D recursive polynomial relations (16), (17) and employs 2-D recusions of constant square matrices and scalars (Mertzios, 1984)

**Algorithm 2**: Two-Dimensional Recursive Inversion Algorithm (TDRIA) of the pencil **A**(s)=(s**E**-**A**).

Initialize

$\mathbf{T}_{00}=\mathbf{I}_n$, $c_{00}=1$ (32)

Boundary conditions

$\mathbf{T}_{-i,j}=\mathbf{T}_{i,-j}=0$, for i>1, j>1 (33a)

$c_{-i,j}=c_{i,-j}=0$ , for i>1, j>1 (33b)

$\mathbf{T}_{i,j}=\mathbf{0}$, for j>i>0 (33c)

$c_{-i,j}=\mathbf{0}$, for j>i>0 (33d)

$\mathbf{T}_{i,j}=\mathbf{0}$, i=f+1, f+2,…,n-1; j=0,1,…i-f-1 (33e)

$\mathbf{c}_{-i,j}=0$ i=r+1, r+2,…,n; j=0,1,…i-r-1 (33f)

2-D recursions

$$\hat{\mathbf{T}}_{i,j} = \mathbf{E}\mathbf{T}_{i-1,j-1} - \mathbf{A}\mathbf{T}_{i-1,j} \tag{34a}$$

$$c_{i,j} = -\frac{1}{i} tr[\hat{\mathbf{T}}_{i,j}], \text{ i=1,2,…,n; j=0,1,…,i} \tag{34b}$$

$$\mathbf{T}_{i,j} = \hat{\mathbf{T}}_{i,j} = c_{i,j}\mathbf{I}_n, \text{ i=1,2,…,n-1; j=0,1,…,i} \tag{34c}$$

Terminate

$\mathbf{B}_i=\mathbf{T}_{n-1,i}$; i=n-1-f, n-f,…,n-1 (35a)

$p_i = c_{n,i}$; i=n-r,n-r+1,…,n (35b)

Final Conditions

$$\mathbf{T}_{n,j} = \hat{\mathbf{T}}_{i,j} + c_{n,j}\mathbf{I}_n = \mathbf{A}\mathbf{T}_{n-1,j} - \mathbf{E}\mathbf{T}_{n-1,j-1} + c_{n,j}\mathbf{I}_n = 0;\ j = 0,1,...,n \qquad (36)$$

**Proof:**

The initial condition (32) results from (15) and(20), (21). The boundary conditions (33a)- (33d) result from the form of (20) and (21).

For the proof of (33e), (33f) we use the fact that $\mathbf{T}_{n-1}$(s)=Adj[-**A**(s)], is a polynomial matrix in s of order f, while $c_n$(s)=det[-**A**(s)] is a polynomial in s of order r. Therefore it is seen from (21)-(23) that the maximum order of $\mathbf{T}_i$(s); i=0,1,…,n-1 is f, and that the maximum order of $c_i$(s); i=0,1,…,n is r.

*3.2 Systems' approach (Fundamental Matrix)*

The CH theorem for regular state space systems in the form (8) will be naturally extended for generalized state space systems. Moreover the fundamental matrix coefficients $\Phi_i$ may be analytically expressed in terms of the systems matrices **E,A**; this relation however is not straightforward and simple as in the regular systems.

Thus the simple and nice expression in terms of $\Phi_i$ may be finally written in terms of **E,A**.

The inverse matrix $A^{-1}$(s) may be written as a Laurent expansion in a deleted neighborhood of zero, as follows:

$$\mathbf{A}^{-1}(s)=(s\mathbf{E}-\mathbf{A})^{-1}=\sum_{i=-\mu}^{\infty}\Phi_i s^{-i-1} = \Phi(s) \qquad (37)$$

where μ is the index of nil potency of the generalized pencil ($s$**E A**) (Langehop, 1979; Rose, 1978) and also the max order of infinite eigenvalues of |$s$**E A**|, i.e. the length of the largest relative eigenvector chain at ∞ (Lewis, 1983). In (37) $\Phi_i$; i=$-\mu,-\mu+1,...,-1,0,1,2,...$ represents the fundamental matrix sequence of (11) and $\Phi\ (s) = L\_[\mathrm{B}\Phi(t)]$ is the resolvent matrix of (37), which is given by the Laplace Trasform of the Fundamental matrix $\Phi(t)$ of (11) (Mertzios and Lewis, 1989).

The inverse matrix $\mathbf{A}^{-1}$(s)=(s**E-A**)$^{-1}$, which is also the resolvent matrix of the singular system (11), may be also expressed as a rational matrix of the form

$$\mathbf{A}^{-1}(s)=(s\mathbf{E}-\mathbf{A})^{-1}=\frac{1}{p(s)}\mathbf{B}(s)=\frac{1}{q(s)}\mathbf{R}(s) \qquad (38)$$

where **B**(s) is the relative adjoint matrix and p(s) is the relative characteristic polynomial of the generalized pencil **A**(s)=(s**E-A**), given by

$$\mathbf{B}(s)=\mathrm{Adj}(s\mathbf{E}-\mathbf{A})=\sum_{k=0}^{f}\mathbf{B}_{n-1-k}s^k=\mathbf{B}_{n-1-f}s^f+\mathbf{B}_{n-f}s^{f-1}+\ldots+\mathbf{B}_{n-2}s+\mathbf{B}_{n-1} \tag{39}$$

$$p(s)=\det(s\mathbf{E}-\mathbf{A})=\sum_{k=0}^{f}p_{n-k}s^k=p_{n-r}s^r+p_{n-r+1}s^{r-1}+\ldots+p_{n-1}s+p_n \tag{40}$$

where $f$ and $r$ are determined by (24) and (25) respectively and

$$\mathbf{R}(s)=\frac{1}{p_{n-r}}\mathbf{B}(s)=\frac{1}{p_{n-r}}\sum_{k=0}^{f}\mathbf{B}_{n-1-k}s^k=\sum_{k=0}^{f}\mathbf{R}_{n-1-k}s^k=$$
$$=\mathbf{R}_{n-1-f}s^f+\mathbf{R}_{n-f}s^{f-1}+\ldots+\mathbf{R}_{n-2}s+\mathbf{R}_{n-1} \tag{41}$$

$$q(s)=\frac{1}{p_{n-r}}p(s)=\frac{1}{p_{n-r}}\sum_{k=0}^{r}p_{n-k}s^k=s^r+\sum_{k=0}^{r-1}q_{n-k}s^k=$$
$$=s^r+q_{n-r+1}+\ldots+s^{r-1}+q_{n-1}s+q_n \tag{42}$$

i.e. both and p(s) **B**(s) have been divided by $p_{n-r}$, which is by construction nonzero. Thus, the leading coefficient $q_{n-r}$, is known and equal to unity.

Finally, equating the right hand sides of (37) and (38) respectively, it results that

$$Q(s)\Phi(s)=\mathbf{R}(s) \tag{43}$$

where q(s), **R**(s), $\Phi$(s) are expressed by (41),(42) and (37) respectively. Equating the coefficient matrices of positive powers of $s$ in the two sides of (43), the following $f$+1 matrix equations result

$$\mathbf{R}_{n-k-1}=\sum_{i=\max\{-\mu,-k\}}^{r-k}q_{n-k-i}\Phi=\sum_{i=\max\{-\mu,-k-1\}}^{r-k-1}q_{n-k-i-1}\Phi_i=$$
$$=\sum_{i=0}^{r}q_{n-k-i-1}\Phi_i=\sum_{i=0}^{r}q_{n-i}\Phi_{i-k-1};k=0,1,\ldots,f \tag{44}$$

where $q_{n-r}$=1. Relations (44) may be written analytically as

$$\mathbf{R}_{n-f+1}=q_{n-r+2}\Phi_{-\mu}+q_{n-r+1}\Phi_{-\mu+1}+\Phi_{-\mu+2} \tag{45a}$$

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$$\mathbf{R}_{n-f+1}=q_{n-r+2}\Phi_{-\mu}+q_{n-r+1}\Phi_{-\mu+1}+\Phi_{-\mu+2} \tag{45a}$$

$$\mathbf{R}_{n-f+\mu-2}=\mathbf{R}_{n-r-1}=q_{n-r+\mu-1}\Phi_{-\mu}+q_{n-r+\mu-2}\Phi_{-\mu+1}+\ldots+\Phi_{-1} \tag{45b}$$

$$\mathbf{R}_{n-r} = q_{n-r+\mu}\Phi_{-\mu} + q_{n-r+\mu-1}\Phi_{-\mu+1} + \ldots + q_{n-r+1} + \Phi_0 \qquad (45c)$$

$$\mathbf{R}_{n-r+1} = q_{n-r+\mu+1}\Phi_{-\mu} + q_{n-r+\mu}\Phi_{-\mu+1} + \ldots + q_{n-r+1} + \Phi_0 + \Phi_1 \qquad (45d)$$

!

$$\mathbf{R}_{n-\mu} = (q_n\Phi_{-\mu} + q_{n-1}\Phi_{-\mu+1} + \ldots + q_{n+\mu+1}\Phi_{-1}) + \\ + (q_{n-\mu}\Phi_0 + q_{n-\mu-1}\Phi_1 + \ldots + q_{n-r+1}\Phi_{r-\mu-1} + \Phi_{r-\mu}) \qquad (45e)$$

$$\mathbf{R}_{n-\mu+1} = (q_n\Phi_{-\mu+1} + q_{n-1}\Phi_{-\mu+2} + \ldots + q_{n-\mu+2}\Phi_{-1}) + \\ + (q_{n-\mu+1}\Phi_0 + q_{n-\mu}\Phi_1 + \ldots + q_{n-r+1}\Phi_{r-\mu} + \Phi_{r-\mu+1}) \qquad (45f)$$

$$\mathbf{R}_{n-1} = (q_n\Phi_{-1}) + (q_{n-1}\Phi_0 + q_{n-2}\Phi_1 + \ldots + q_{n-r+1}\Phi_{r-2} + \Phi_{r-1}) \qquad (45g)$$

In (45) $\mathbf{R}_{n-k-1}$, $\Phi_{i-k-1}$,$q_{n-i}$ are coefficients of $s^k$, $s^{k-i}$, $s^i$ in the polynomial matrices **R**(s), $\Phi$(s) and the polynomial q(s) respectively. Moreover, equating the coefficient matrices of negative powers of s in (43), the following ARMA matrix equation results

$$\mathbf{R}_{n+k} = \sum_{i=0}^{r} q_{n-1}\Phi_{k+i} = \Phi_{\mathbf{r+k}} + q_{n-r+1}\Phi_{n+k+1} + \ldots + q_{n-1}\Phi_{k+1} + q_n\Phi_k = 0, \text{ for } k \geq 0 \qquad (46)$$

The relations (46) represent the relative Caley-Hamilton (CH) Theorem for generalized systems (Mertzios 1983; Lewis 1986).

The fundamental matrix sequence $\Phi_i$ may be calculated using explicit relations in terms of the system's matrices **E,A** (Mertzios and Lewis, 1989).

Specifically it is sufficient to calculate only $\Phi_{-1}$ and $\Phi_0$ while $\Phi_i$; i=-2,-3,…,$-\mu$ and $\Phi_i$; i=1,2,… are recursively calculated in terms of $\Phi_i$, and $\Phi_0$ respectively.

## 4. THE CAYLEY-HAMILTON THEOREM IN MATRIX DIFFERENTIAL SYSTEMS

The Matrix Differential Systems (MDSs), or ARMA (Autoregressive-Moving-Average) systems are dynamical systems of the (Willems, 1991; Baser and Schumacher, 2000; Campbel and Campbel and Schumacher, 2002; Vidyasagar, 1985)

$$\mathbf{A}(D)\mathbf{x}(t)=\mathbf{B}(D)\mathbf{u}(t) \qquad (47a)$$

$$\mathbf{y}(t)=\mathbf{C}(D)\mathbf{x}(t)+\mathbf{D}(D)\mathbf{u}(t) \qquad (47b)$$

where $\mathbf{x}(t) \in \Re^n$, $\mathbf{u}(t) \in \Re^m$, $\mathbf{y}(t) \in \Re^1$, $\mathbf{D}=\dfrac{d}{dt}$, $\mathbf{A}(D) \in \Re[D]^{nxn}, \det[\mathbf{A}(D)] \neq 0$,

$\mathbf{B}(D) \in \Re[D]^{nxm}, \mathbf{C}(D) \in \Re[D]^{lxn}, \mathbf{D}(D) \in \Re[D]^{lxm}$ and may be considered as a generalization of the generalized state-space model (11).

*4.1 Algebraic approach (Characteristic Equation).*

The resolvent matrix of (47) is the inverse matrix $\mathbf{A}^{-1}(s)$, where $\mathbf{A}(s)$ is the square polynomial matrix

$$\mathbf{A}(s)=s^k\mathbf{A}_k+\ldots+s\mathbf{A}_1+\mathbf{A}_0 \in \Re[s]^{nxn} \tag{48}$$

Then the characteristic equation of is $\mathbf{A}(s)$ is

$$P(s,z)=\det\,[z\mathbf{I}_n\text{-}\mathbf{A}(s)]=p_0(s)z^n+p_1(s)z^{n-1}+\ldots+p_{n-1}(s)z+p_n(s)=0 \tag{49}$$

where $p_i(s)$; $i=1,2,\ldots,n$ are the coefficient polynomials appearing in the two-variable characteristic polynomial $p(s,z)$ of $\mathbf{A}(s)$.

Extending the underlying idea of the Cayley-Hamilton Theorem, $\mathbf{A}(s) \in \Re^{nxn}$, satisfies its characteristic equation (Mertzios et.al., 1986)

$$q[s,\mathbf{A}(s)=q_0(s)\mathbf{A}^n(s)+q_1(s)\mathbf{A}^n(s)+\ldots+q_{n-1}(s)\mathbf{A}(s)+q_n(s)\mathbf{I}_n=0 \tag{50}$$

for all s, which represents a polynomial matrix equal to zero.

*4.2 Systems' approach (Fundamental Matrix)*

The inverse matrix $\mathbf{A}^{-1}(s)$, in the convolutional type of notation, are written as

$$\mathbf{A}^{-1}(s)=(s^k\mathbf{A}_k+\ldots+s\mathbf{A}_1+\mathbf{A}_0)^{-1}=\frac{1}{p(s)}\mathbf{B}(s)\frac{1}{\sum_{m=0}^{r} p_{r-m}s^m}[\sum_{m=0}^{f} \mathbf{B}_{f-m}s^m] \tag{51}$$

where

$$r=\deg p(s) \leq f = \deg \mathbf{B}(s) \leq (n-1)k \tag{52}$$

Moreover, in general $\mathbf{A}^{-1}(s)$ has a Laurent expansion of the form

$$\mathbf{A}^{-1}(s)=(s_k\mathbf{A}^k+\ldots+s\mathbf{A}_1+\mathbf{A}_0)^{-1}=\sum_{i=-\mu}^{\infty}\Phi_i s^{-i-1} \tag{53}$$

where $\mu \geq 1$, is the index of nil potency of $\mathbf{A}(s)$ at $s=\infty$ at the Smith-McMillan from (Fragulis et. Al., 1991).

Equating the right-hand side parts of (51) and (53), I obtain:

$$\mathbf{B}(s) = \sum_{m=0}^{f} \mathbf{B}_{f-m} s^{m} = [\sum_{m=-\mu}^{\infty} \Phi_m s^{-m-1}] \quad (54)$$

from which it is seen that f=r+ $\mu - 1$ and the following relations result:

$$\mathbf{B}_m = \sum_{j=0}^{\min\{r,m\}} p_j \Phi_{-\mu+m-j}, \text{ for m=0,1,…,f} \quad (55)$$

$$0 = \sum_{j=0}^{r} p_j \Phi_{-\mu+m-j}, \; for \;\; m \ge f+1$$

$$0 = \sum_{j=0}^{r} p_j \Phi_{-\mu+m-j}, \; for \;\; m \ge 0 \quad (56)$$

Note that (54) represents the CH Theorem, which is written in the form (8) (Lewis, 1986).

## 5. CONCLUSIONS

Both the described approaches for the extraction of the Cayley-Hamilton theorem for the generalized and matrix differential systems may be extended to multidimensional systems, as well as to any other implicit state-space systems.

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**Author:**

**Basil G. Mertzios,** Department of Automation, Thessaloniki Institute of Technology, Thessaloniki, Hellas, **mertzios@uom.gr**.