TWO STARLIKENESS PROPERTIES FOR THE BERNARDI OPERATORS

by
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Abstract. It is well known that if f is starlike, then F is also starlike.In this paper we extend this result if f satisfies larger conditions and proving that F is starlike of order $2/3$ and $1/3$, where F is the Bernardi integral operator.

Introductions.

Let $U = \{z : |z| < 1\}$ the unit disc in the complex plane, and let $H(U)$ the space of holomorphic functions in *U*.
Let A_n denote the class of functions f of the form

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + ..., z \in U$$

wich are analytic in the unit disc U and $A_1 = A$.
Let the class of univalent functions $S = \{f : f \in H_u(U), f(0) = f'(0) - 1 = 0\}$ and let

$$S^*(\alpha) = \left\{ f : \mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha, z \in U \right\}$$

be the class of the starlike functions of the order $\alpha, 0 \le \alpha < 1$, in *U*. For $\alpha = 0$, denote $S^*(0) = S^*$ the class of the starlike functions in *U*.
Consider the following integral operator

$$F(z) = \frac{\gamma + 1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt,\ \gamma \ge 0,$$

where $f \in A_n$.

Lemma A.[1] Let q univalent in U and let θ and ϕ analytic functions in the domain $D \subseteq q(U)$, with $\phi(w) \ne 0$, when $w \in q(U)$.

Let

$$Q(z) = nzq'(z)\phi[q(z)] \; h(z) = \theta[q(z)] + Q(z)$$

and suppose that next condition is satisfy:

i) Q is starlike

and

ii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} = \operatorname{Re}\left[\dfrac{\theta'[q(z)]}{\phi[q(z)]} + \dfrac{zQ'(z)}{Q(z)}\right] > 0$.

If p is analytic in U, with

$$p(0) = q(0), \quad p'(0) = \ldots = p^{(n-1)}(0) = 0, \quad p(U) \subset D$$

and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)] = h(z)$$

then $p \prec q$ and q is the best dominant.

Main results.

Theorem 1. Let

$$h(z) = \frac{2}{2-z} + \frac{2nz}{(2-z)(2+2\gamma-\gamma z)}$$

If $f \in A_n$ and

$$\frac{zf'(z)}{f(z)} \prec h(z),$$

then

$$\operatorname{Re} \frac{zF'(z)}{F(z)} > \frac{2}{3},$$

where

$$F(z) = \frac{1+\gamma}{z^{\gamma}} \int_0^z f(t) t^{\gamma-1} dt \quad (1)$$

Proof. The relation (1) implies:

$$\gamma F(z)+zF'(z)=(\gamma+1)f(z) \qquad (2)$$

Let

$$p(z)=\frac{zF'(z)}{F(z)}.$$

The relation (2) implies

$$\frac{zp'(z)}{p(z)+\gamma}+p(z)=\frac{zf'(z)}{f(z)}.$$

But

$$\frac{zf'(z)}{f(z)}\prec h(z)$$

implies

$$\frac{zp'(z)}{p(z)+\gamma}+p(z)\prec h(z).$$

We apply the lemma A for proved that

$$\operatorname{Re}\frac{zF'(z)}{F(z)}>\frac{2}{3}.$$

If we let:

$$q(z)=\frac{2}{2-z}$$

$$\theta(w)=w$$

$$\phi(w)=\frac{1}{w+\gamma}$$

$$\theta[q(z)]=\frac{2}{2-z}$$

$$\phi[q(z)]=\frac{2-z}{2+2\gamma-\gamma z}$$

$$Q(z) = nzq'(z)\phi[q(z)] = \frac{2nz}{(2-z)(2+2\gamma-\gamma z)}.$$

$$h(z) = \theta[q(z)] + Q(z) = \frac{2}{2-z} + \frac{2nz}{(2-z)(2+2\gamma-\gamma z)}.$$

Since Q is starlike and $\operatorname{Re}\phi[q(z)] > 0$, by lemma A we deduce

$$p \prec q \Leftrightarrow \frac{zF'(z)}{F(z)} \prec \frac{2}{2-z} \Rightarrow \operatorname{Re}\frac{zF'(z)}{F(z)} > \frac{2}{3}.$$

Theorem 2. Let

$$h(z) = \frac{2+z}{2-z} + \frac{4nz}{(2-z)(2+2\gamma+(1-\gamma)z)}.$$

If $f \in A_n$ and

$$\frac{zf'(z)}{f(z)} \prec h(z),$$

then

$$\operatorname{Re}\frac{zF'(z)}{F(z)} > \frac{1}{3},$$

where

$$F(z) = \frac{1+\gamma}{z^{\gamma}} \int_0^z f(t)t^{\gamma-1}dt.$$

Proof. We are

$$\gamma F(z) + zF'(z) = (\gamma+1)f(z) \qquad (3)$$

Let

$$p(z) = \frac{zF'(z)}{F(z)}.$$

We deduce

$$\frac{zp'(z)}{p(z)+\gamma} + p(z) = \frac{zf'(z)}{f(z)}.$$

But

$$\frac{zf'(z)}{f(z)} \prec h(z)$$

and obtained

$$\frac{zp'(z)}{p(z)+\gamma}+p(z)\prec h(z).$$

In lemma we consider:

$$q(z)=\frac{2+z}{2-z}$$

$$\theta(w)=w$$

$$\phi(w)=\frac{1}{w+\gamma}$$

$$\theta[q(z)]=\frac{2+z}{2-z}$$

$$\phi[q(z)]=\frac{2-z}{2+2\gamma+(1-\gamma)z}.$$

$$Q(z)=nzq'(z)\phi[q(z)]=\frac{2nz}{(2-z)(2+2\gamma+(1-\gamma)z)}$$

$$h(z)=\theta[q(z)]+Q(z)=\frac{2+z}{2-z}+\frac{2nz}{(2-z)(2+2\gamma+(1-\gamma)z)}.$$

Sience Q is starlike and $\operatorname{Re}\phi[q(z)]>0$, by lemma A obtained

$$p\prec q\Leftrightarrow\frac{zF'(z)}{F(z)}\prec\frac{1+z}{1-z}\Rightarrow\operatorname{Re}\frac{zF'(z)}{F(z)}>\frac{1}{3}$$

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