MATRIX ALGEBRAS OVER GRASSMANN ALGEBRAS AND THEIR PI-STRUCTURE

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Abstract. In the paper we explore the PI-structure of some algebras of upper triangular matrices over Grassmann algebras. Applying the tensor product construction for the Grassmann algebra $G(V)$ and two matrix algebras we illustrate that the corresponding tensor products satisfy the Grassmann identity $[x_1, x_2, x_3] = 0$ as well. Considering some noncommutative matrix algebras over concrete finite dimensional nonunitary Grassmann algebras $G'(V_n)$ for small n we define a natural k such that the identity $X^k = 0$ holds in the corresponding matrix algebras over $G'(V_n)$.

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1. Preliminaries

We consider some algebras of upper triangular matrices over the infinite dimensional Grassmann algebra G and over the nonunitary finite dimensional Grassmann algebras G'_n for $n = 4, 5, 6$ discussing the case of arbitrary n as well. The algebra G is defined as

$$G = G(V) = K\langle e_1, e_2, \dots \mid e_ie_j + e_je_i = 0 \ i, j = 1, 2, \dots\rangle.$$

The field K has a characteristic zero. The algebra G' (without 1) has a basis $\{e_{i_1}e_{i_2}\dots e_{i_k}\}$, where $1 \le i_1 < i_2 \dots < i_k$. The elements e_i are called generators of G' while the elements $e_{i_1}e_{i_2}\dots e_{i_k}$ for $1 \le i_1 < i_2 \dots < i_k$ are called basic monomials of G'. For $G = G' \cup K$ the element 1 is a generator as well. The algebras G and G' are PI-equivalent (they satisfy one and the same identities).

The algebra G is in the mainstream of recent research in PI-theory. Its importance is connected with the structure theory for the T-ideals of identities of associative algebras developed by Kemer. In [10, Theorem 1.2] he proved that any T-prime T-ideal can be obtained as the T-ideal of identities of one of the following algebras: $M_n(K)$, $M_n(G)$ and $M_{n,u}(G)$, the latter being the algebra of $n \times n$ supermatrices over $G = G_0 \oplus G_1$ with two G_0 blocks (with entries of even degree) of sizes $u \times u$ and $(n-u) \times (n-u)$ and with two G_1 blocks (with entries of odd degree) of sizes $u \times (n-u)$ and $(n-u) \times u$.

Another reason for the Grassmann algebra to be one of the fundamental structures in PI-theory is the fact that it generates a minimal variety of exponential growth [11].

There is a motivation of considering finite-dimensional Grassmann algebras as well and it is connected with the emergence in mathematical physics mainly in quantum field theory of methods from algebraic geometry and Grassmann algebras. We give only three examples here:

If we take a Grassmann algebra with two generators y and y^* and a conjugation $*$ we have $(y^*)^* = y$ and one could define Grassmann differentiation and integration, the exponential function, scalar product of linear functions, etc. Thus for example Fermion coherent states for Bosons could be introduced and its physical significance investigated [7].

In [15] Schornhorst considers a special type integral equation with an unknown function over a finite dimensional Grassmann algebra G_{2n} and gives conditions for the existence of solutions of this equation for $n = 2$ and $n = 4$. The choice of the equation is motivated by the effective action formalism of lattice quantum field theory.

Concerning nonrelativistic theory functions on phase space are elements of a Grassmann algebra with three generators [3].

The importance of considering matrix algebras $M_n(G)$ is confirmed by the following statement as the trivial isomorhism $G \otimes M_n(K) \simeq M_n(G)$ holds:

Proposition 1 [6, Corollary 8.2.4, p. 111] *For every PI-algebra R there exists a positive n such that $T(R) \supseteq T(M_n(G))$, i.e. R satisfies all polynomial identities of the $n \times n$ matrix algebra $M_n(G)$ with entries from the Grassmann algebra.*

Some well known facts concerning the algebra G are the following:

Proposition 2 [11, Corollary, p. 437] *The T-ideal $T(G)$ is generated by the identity $[x_1, x_2, x_3] = 0$.*

Proposition 3 [2, Lemma 6.1] *The algebra G satisfies $S_n(x_1, \ldots, x_n)^k = 0$*

for all $n, k \geq 2$ *and*

$$S_n(x_1, \dots, x_n) = \sum_{\sigma \in Sym(n)} (-1)^\sigma x_{\sigma(1)} \dots x_{\sigma(n)}$$

being the standard identity.

Proposition 4 [5, Exercise 5.3] *For* $G_k = G(V_k)$ *over* k*-dimensional vector space* V_k *all identities follow from the identity* $[x_1, x_2, x_3] = 0$ *and the standard identity*

$$S_{2p}(x_1, \dots, x_{2p}) = 0,$$

where p *is the minimal integer such that* $2p > k$*.*

Proposition 5 [8, Theorem 3.5] *Let* K *be an infinite field. A basis of the identities of* G_{2k} *is given by the polynomials*

$$[x_1, x_2, x_3] = 0, \ [x_1, x_2] \dots [x_{2k+1}, x_{2k+2}] = 0.$$

The identities for an algebra are connected with its central polynomials. The polynomial $c(x_1, ..., x_m)$ from the free associative algebra $K\langle X\rangle$ is called a *central polynomial* for an algebra R if it has no constant term, $c(r_1, ..., r_m)$ belongs to the centre of R for all $r_1, ..., r_m \in R$ and $c(x_1, ..., x_m) = 0$ is not a polynomial identity for R.

It is interesting to note that the polynomial $c(x_1, x_2) = [x_1, x_2]$ is a central polynomial for the algebra G of minimal degree and this degree is minimal in general if we consider as trivial the case of commutative algebras for which the polynomial x is central.

Proposition 6 [4, Proposition 5] *The vector space of the central polynomials of* G *is generated (as a* T*-space in* $K\langle X\rangle$*) by* 1 *and the polynomials* $x_1[x_2, x_3, x_4]$ *and* $[x_1, x_2]$*.*

2. Identities for $M_n(G)$ and $U_n(G)$

The algebra $U_n(K)$ of the upper triangular $n \times n$ matrices over the field K plays an important role in PI-theory. The identities of $U_n(K)$ describe in certain sense the subvarieties of the variety of algebras generated by the matrix algebra of order two.

Following this law we state here the known identities both for the algebras $M_n(G)$ and $U_n(G)$.

Proposition 7 [14, Theorem] *The matrix algebra $M_n(G)$ has no identities of degree $4n-2$.*

Vishne gave in [17] explicit identities of degree 8 for $M_2(G)$ and concluded the following

Proposition 8 [17, Corollary 4.5] *If n is even the degree of a multilinear identity for $M_n(G)$ is at least $4n$.*

The identity of "algebraicity" for matrices over the Grassmann algebra was defined by J. Szigeti in [16].

Proposition 9 [16, Theorem 5.1.] *The polynomial*

$$S_{2n^2}([X^{2n^2},Y],[X^{2n^2-1},Y],\ldots,[X^2,Y],[X,Y])=0$$

is an identity for $M_n(G)$.

In the case of upper triangular matrices we cite a well known fact, following from [9, Theorem 1.9.1], namely

Proposition 10 *The identity $[X,Y,Z]^n=0$ holds for any three upper triangular matrices X,Y,Z from $U_n(G)$.*

For any PI-algebra R we denote by $c_m(R)$ the dimension of the space of the multilinear polynomials of degree m modulo the T-ideal $T(R)$ and define the exponent $exp(R)=\lim_{m\to\infty}\sqrt[m]{c_m(R)}$. The exponent of a PI-algebra can serve as a scale for the complexity of the polynomial identities. If $T(R_1)\subset T(R_2)$, then R_2 has more identities than R_1 and $exp(R_1)\geq exp(R_2)$. Due to Krakowski, Regev and Berele it is known that $c_m(G)=2^{m-1}$ and $exp(M_n(G))=2n^2$ as cited in [6, p.19, 114]. In [8] it was proved that $c_m(G_{2k})=\sum_{j=0}^{k}\binom{m}{2j}$

3. Examples of matrix algebras over G that satisfy the Grassmann identity

As Proposition 10 holds we are interested in finding some classes of upper triangular matrices $U_n(G)$ for arbitrary n for which a lower degree (not dependent of n) of the triple commutator (of length 3) is an identity. The tensor product construction of G with commutative matrix algebras leads to the next two propositions.

For simplicity when working in concrete 2×2 or 3×3 matrix algebras we'll denote the Grassmann elements by Greek letters. In the statements for $n \times n$ matrices for arbitrary n we'll use the roman notation for the Grassmann elements.

Proposition 11 *The algebra*

$$U1(G) = \{ \begin{pmatrix} a & a_{12} & a_{13} & \dots & \dots & a_{1n-1} & a_{1n} \\ 0 & a & 0 & \dots & \dots & 0 & a_{1n-1} \\ 0 & 0 & a & 0 & \dots & 0 & a_{1n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & a & a_{12} \\ 0 & \dots & \dots & \dots & \dots & 0 & a \end{pmatrix} | a, a_{1i} \in G, i = 2, .., n\}$$

satisfies the Grassmann identity $[X_1, X_2, X_3] = 0$.

Proof: We give a direct proof of the proposition. We present any matrix X_s (where $1 \leq s \leq 3$) as a sum $X_s = Y_s + Z_s$ for $Y_s = \sum_{i=1}^{n} a_s e_{ii} + \alpha_s e_{1n}$ and $Z_s = a_{12}^{(s)}(e_{12} + e_{n-1\,n}) + a_{13}^{(s)}(e_{13} + e_{n-2\,n}) + a_{14}^{(s)}(e_{14} + e_{n-3\,n}) + \dots + a_{1n-1}^{(s)}(e_{1n-1} + e_{2n})$. If in the expression considered there is only one matrix the index "s" will be omitted.

For the corresponding triple commutators we get
$[Y_1, Y_2, Y_3] = [a_1, a_2, a_3] \sum_{i=1}^{n} e_{ii} + ([a_1, a_2, \alpha_3] + [a_1, \alpha_2, a_3] + [\alpha_1, a_2, a_3])e_{1n}$;
$[Y_1, Y_2, Z] = [a_1, a_2, a_{12}](e_{12} + e_{n-1\,n}) + [a_1, a_2, a_{13}](e_{13} + e_{n-2\,n}) +$
$\dots + [a_1, a_2, a_{1\,n-1}](e_{1n-1} + e_{2n})$;
$[Y_1, Z, Y_2] = [a_1, a_{12}, a_2](e_{12} + e_{n-1\,n}) + [a_1, a_{13}, a_2](e_{13} + e_{n-2\,n}) +$
$\dots + [a_1, a_{1n-1}, a_2](e_{1n-1} + e_{2n})$;
$Z_1 Z_2 Z_3 \equiv 0$;
$[Z_1, Z_2, Y] = ([a_{12}^{(1)}, a_{1n-1}^{(2)}, a] + [a_{13}^{(1)}, a_{1n-2}^{(2)}, a] + \dots + [a_{1n-1}^{(1)}, a_{12}^{(2)}, a])e_{1n}$;
$[Y, Z_1, Z_2] = ([a, a_{12}^{(1)}, a_{1n-1}^{(2)}] + [a, a_{13}^{(1)}, a_{1n-2}^{(2)}] + \dots + [a, a_{1n-1}^{(1)}, a_{12}^{(2)}])e_{1n}$.

Proposition 2 gives that all these commutators are zero. As $[X_1, X_2, X_3] = [Y_1 + Z_1, Y_2 + Z_2, Y_3 + Z_3]$ we get $[X_1, X_2, X_3] = 0$.

Corrolary 1 *For the algebra* $U1(G)$ *the polynomial* $[X_1, X_2]$ *is central and* $c_m(U1(G)) = 2^{m-1}$.

Another example gives

Proposition 12 *The algebra*

$$U2(G) = \{\begin{pmatrix} a & a_{12} & a_{13} & ... & ... & a_{1n-1} & a_{1n} \\ 0 & a & 0 & ... & ... & 0 & -a_{1n-1} \\ 0 & 0 & a & 0 & ... & 0 & -a_{1n-2} \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & ... & ... & ... & a & -a_{12} \\ 0 & ... & ... & ... & ... & 0 & a \end{pmatrix} |a, a_{1i} \in G, i = 2,..,n\}$$

satisfies the Grassmann identity $[X_1, X_2, X_3] = 0$.

4. Nilpotency of the elements of some matrix algebras over G'_n for $n = 4, 5, 6$

Very little is known for the identities of $M_n(G)$ even for $n = 2$ (except the Vishne identities [17]). Thus investigations seem to be useful even for special matrix subalgebras and even over finite dimensional Grassmann algebras. In the next investigations we deal with concrete nonunitary finite dimensional Grassmann algebras G'_n. Our aim is to find such classes of upper triangular $s \times s$ algebras over G'_n for which there exists a natural $k < n + 1 : X^k = 0$ is an identity for them. As G'_n are nilpotent algebras of index $\leq n + 1$ the same is valid for $M_s(G'_n)$ and obviously $X^{n+1} = 0$ for all such algebras.

Using a programme written in the system for computer algebra *Mathematica* [13] we find $m < n + 1$ such that $x^m = 0$ holds for the considered finite dimensional Grassmann algebras G'_n:

Proposition 13 *The identity* $x^3 = 0$ *holds for the algebra* G'_4.

Proposition 14 *The identity* $x^4 = 0$ *holds for the algebras* G'_5 *and* G'_6.

The algebras of upper triangular matrices that we consider here are provoked by [12]. The authors there study the $*$-varieties of associative algebras with involution $*$ over a field of characteristic zero which are generated by a finite-dimensional algebra. Mattina and Misso gave a list of algebras classifying all such $*$-varieties whose sequence of $*$-codimensions is linearly bounded (these are the dimensions of the space of multilinear polynomials in n $*$-variables for $n = 1, 2, ...$ in the corresponding relatively free algebra with involution of countable rank).

Matrix algebras over G_4'.

Proposition 15 *In G_4' the following identities hold:*

$$\begin{aligned}&\beta\alpha^2+\alpha^2\beta=0,\\&\alpha\beta\alpha=0,\\&\alpha\gamma\beta+\beta\gamma\alpha=0,\\&\alpha\beta\gamma\delta=-\alpha\beta\delta\gamma.\end{aligned}$$

Proof: The partial linearization of $\alpha^3=0$ is $\beta\alpha^2+\alpha\beta\alpha+\alpha^2\beta=0$. The Grassmann identity $[\beta,\alpha,\alpha]=0$ gives $\beta\alpha^2-\alpha\beta\alpha-\alpha\beta\alpha+\alpha^2\beta=0$. As the field K is of characteristic 0 we get the first two identities. The linearization of the second identity gives the third one. Applying the third identity 3 times we get the last one, namely $\alpha\beta\gamma\delta=-\gamma\beta\alpha\delta=\gamma\delta\alpha\beta=-\alpha\beta\delta\gamma$.

Proposition 16 *The algebra $M_2(G_4')$ satisfies the identity $X^4=0$.*

Proof: For $A=\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}$ we get that

$$A^4=\begin{pmatrix}\beta\delta\gamma\alpha+\alpha\beta\delta\gamma+\beta\delta^2\gamma & \alpha^2\beta\delta+\alpha\beta\gamma^2\\ \delta\gamma\alpha^2+\delta^2\gamma\alpha & \gamma\alpha^2\beta+\delta\gamma\alpha\beta+\gamma\alpha\beta\delta\end{pmatrix}.$$

According to Proposition 15

$$\begin{aligned}&\alpha^2\beta\gamma=\alpha\beta^2\gamma=\alpha\beta\gamma^2=0,\\&\beta\delta\gamma\alpha=-\beta\delta\alpha\gamma=\gamma\delta\alpha\beta=-\alpha\beta\delta\gamma,\\&\delta\gamma\alpha\beta=-\alpha\beta\gamma\delta=\gamma\delta\beta\alpha=-\gamma\alpha\beta\delta.\end{aligned}$$

Thus we see that all entries of the matrix A^4 are zero.

Proposition 17 *The matrix algebras $U3(G_4')=\{\begin{pmatrix}\alpha&\delta&0\\0&\beta&0\\0&0&\gamma\end{pmatrix}\}$, $U4(G_4')=\{\begin{pmatrix}\alpha&0&\delta\\0&\beta&0\\0&0&\gamma\end{pmatrix}\}$ and $U5(G_4')=\{\begin{pmatrix}\alpha&0&0\\0&\beta&\delta\\0&0&\gamma\end{pmatrix}\}$ satisfy the identity $X^4=0$.*

Proof: For $A \in U3(G_4')$ modulo $\kappa^3 = 0$ for any $\kappa \in G_4'$ we get that the only nontrivial entry of A^4 is the $(1,2)$ entry,namely $\alpha^2\delta\beta + \alpha\delta\beta^2$. According to Proposition 15 it is zero.

Analogous are the other two cases.

Using the same direct approach we get

Proposition 18 *The matrix algebras* $U6(G_4') = \{\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix}\}$, $U7(G_4') = \{\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & 0 \end{pmatrix}\}$ *and* $U8(G_4') = \{\begin{pmatrix} 0 & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & 0 \end{pmatrix}\}$ *satisfy the identity* $X^4 = 0$.

The identity $X^4 = 0$ is satisfied as well by some upper triangular $n \times n$ matrices over G_4' for arbitrary n. We formulate

Theorem 1 *The matrix algebras* $U9(G_4') = \{\begin{pmatrix} 0 & \dots & \dots & \dots & 0 & a_{1n-1} & a_{1n} \\ 0 & a & 0 & \dots & \dots & 0 & a_{2n} \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & a & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & 0 \end{pmatrix}\}$

and $U10(G_4') = \{\begin{pmatrix} a & a_{12} & a_{13} & \dots & \dots & a_{1n-1} & a_{1n} \\ 0 & \dots & \dots & \dots & \dots & 0 & a_{2n} \\ 0 & \dots & \dots & \dots & \dots & 0 & a_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & 0 & a_{n-1n} \\ 0 & \dots & \dots & \dots & \dots & 0 & a \end{pmatrix}\}$ *satisfy the identity* $X^4 = 0$.

Proof: For $A \in U9(G_4')$ we get that modulo $a^3 = 0$ the only non-zero entries of $A^3 = B$ are $b_{1n-1} = a_{1n-1}a^2$ and $b_{2n} = a^2 a_{2n}$. Thus $A^4 = 0$ modulo $a^3 = 0$.

Let A be an element of the algebra $U10(G_4')$. We follow the $(1,n)$-entry in the powers of A:

For $A^2 = B$ we get $b_{1n} = aa_{1n} + a_{12}a_{2n} + \cdots + a_{1n}a$.

For $A^3 = C$ we have

$$c_{1n} = a^2 a_{1n} + aa_{12}a_{2n} + \cdots + aa_{1n-1}a_{n-1n} + aa_{1n}a + a_{12}a_{2n}a + \cdots + a_{1n}a^2.$$

Applying Proposition 15 we get

$$c_{1n} = aa_{12}a_{2n} + \cdots + aa_{1n-1}a_{n-1n} + a_{12}a_{2n}a + \cdots + a_{1n-1}a_{n-1n}a.$$

For $A^4 = D$ we define the corresponding entry, namely

$$d_{1n} = a^2 a_{12}a_{2n} + a^2 a_{13}a_{3n} + \cdots + a^2 a_{1n-1}a_{n-1n} +$$
$$aa_{12}a_{2n}a + aa_{13}a_{3n}a + \cdots + aa_{1n-1}a_{n-1n}a + a_{12}a_{2n}a^2 + \cdots + a_{1n-1}a_{n-1n}a^2.$$

Applying Proposition 15 the middle $(n-2)$-th summands are zero. Then we group the first of the remaining summands with the $(n-1)$-th one, the second with the n-th and so on. All these $(n-2)$ groups are zero using Proposition 15.

All other entries include a^3 and applying Preposition 13 we get that $A^4 = 0$.

Matrix algebras over G_5'.

Proposition 19 *In G_5' the following identities hold:*

$$\alpha^3\beta + \beta\alpha^3 = 0,$$
$$\alpha\beta\alpha^2 + \alpha^2\beta\alpha = 0,$$
$$\alpha^2\beta\alpha^2 = 0.$$

Proof: The partial linearization of $\alpha^4 = 0$ gives $\beta\alpha^3 + \alpha\beta\alpha^2 + \alpha^2\beta\alpha + \alpha^3\beta = 0$. The Grassmann identity $[\beta, \alpha, \alpha] = 0$ could be written as $\beta\alpha^2 + \alpha^2\beta = 2\alpha\beta\alpha$. Multiplying it once by α on the left, then by α on the right and summing we get that modulo the above partial linearization $\alpha\beta\alpha^2 + \alpha^2\beta\alpha = 0$ and thus $\alpha^3\beta + \beta\alpha^3 = 0$. The Grassmann identity $[\alpha^2, \beta, \alpha^2] = 0$ gives (modulo $\alpha^4 = 0$) that $\alpha^2\beta\alpha^2 = 0$.

Proposition 20 *The matrix algebras* $U11(G_5') = \{\begin{pmatrix} 0 & \beta & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}\}$, $U12(G_5') = \{\begin{pmatrix} 0 & 0 & \beta \\ 0 & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}\}$ *and* $U13(G_5') = \{\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & 0 \end{pmatrix}\}$ *satisfy the identity* $X^5 = 0$

Theorem 2 *The matrix algebras* $U14(G_5') = \{\begin{pmatrix} 0 & ... & ... & ... & 0 & a_{1n-1} & a_{1n} \\ 0 & a & 0 & ... & ... & 0 & a_{2n} \\ 0 & ... & ... & ... & ... & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & ... & ... & 0 & a & 0 \\ 0 & ... & ... & ... & ... & 0 & 0 \end{pmatrix}\}$ *and* $U15(G_5') = \{\begin{pmatrix} a & a_{12} & a_{13} & ... & ... & a_{1n-1} & a_{1n} \\ 0 & 0 & 0 & ... & ... & 0 & a_{2n} \\ 0 & 0 & 0 & 0 & ... & 0 & a_{3n} \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & ... & ... & ... & 0 & a_{n-1n} \\ 0 & ... & ... & ... & ... & 0 & a \end{pmatrix}\}$ *satisfy the identity* $X^5 = 0$.

Proof: The proof follows the same pattern as the proof of Theorem 1. For any A of $U14(G_5')$ (of $U15(G_5')$) we form A^2, A^3, $A^4 = B$. We define b_{1n}, apply Proposition 19 and get that $A^5 = 0$.

Matrix algebras over G_6'.

As Proposition 19 holds for G_6' as well analogues of Proposition 20 and Theorem 2 hold.

Proposition 21 *The matrix algebra* $U16(G_6') = \{\begin{pmatrix} 0 & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & 0 \end{pmatrix}\}$ *satisfies the identity* $X^6 = 0$.

Proof: For the matrix $A = \begin{pmatrix} 0 & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & 0 \end{pmatrix}$ modulo $\alpha^4 = 0$ we get that the only nonzero entry of A^5 is the $(1,3)$-entry, equal to $\beta\alpha^3\beta$. Thus $A^6 = 0$.

5. The identity $x^k = 0$

Theorem 1 and Theorem 2 could be generalized in an obvious way, namely

Theorem 3 *Let the identities* $x^k = 0$ *for a given* k *and* $[x_1, x_2, x_3] = 0$ *hold in the associative algebra GR over a field of characteristic zero. Then for the* *matrix algebras* $U17(GR) = \{ \begin{pmatrix} 0 & ... & ... & ... & 0 & a_{1n-1} & a_{1n} \\ 0 & a & 0 & ... & ... & 0 & a_{2n} \\ 0 & ... & ... & ... & ... & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & ... & ... & 0 & a & 0 \\ 0 & ... & ... & ... & ... & 0 & 0 \end{pmatrix} \}$ *and*

$U18(GR) = \{ \begin{pmatrix} a & a_{12} & a_{13} & ... & ... & a_{1n-1} & a_{1n} \\ 0 & ... & ... & ... & ... & 0 & a_{2n} \\ 0 & ... & ... & ... & ... & 0 & a_{3n} \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & ... & ... & ... & 0 & a_{n-1n} \\ 0 & ... & ... & ... & ... & 0 & a \end{pmatrix} \}$ *the identity* $X^{k+1} = 0$ *holds.*

Before proving the theorem we need analogues of Proposition 15 and Proposition 19, namely

Proposition 22 *Let the associative algebra* GR *satisfy the identities* $x^k = 0$ *for a given* k *and* $[x_1, x_2, x_3] = 0$. *Then the following identities hold, namely, for* $k = 2l + 1$

$$\begin{aligned} \alpha x^{k-1} + x^{k-1}\alpha &= 0, \\ x\alpha x^{k-2} + x^{k-2}\alpha x &= 0, \\ &... \\ x^{l-1}\alpha x^{l+1} + x^{l+1}\alpha x^{l-1} &= 0, \\ x^l \alpha x^l &= 0 \end{aligned}$$

and for $k = 2l$

$$\begin{aligned} \alpha x^{k-1} + x^{k-1}\alpha &= 0, \\ x\alpha x^{k-2} + x^{k-2}\alpha x &= 0, \\ &... \\ x^{l-1}\alpha x^{l} + x^{l}\alpha x^{l-1} &= 0. \end{aligned}$$

Proof: We'll consider only the case when k is odd, namely $k = 2l + 1$. The partial linearization of $x^k = 0$ gives the identity

$$\alpha x^{k-1} + x\alpha x^{k-2} + x^2\alpha x^{n-3} + \cdots + x^{k-1}\alpha = 0. \tag{1}$$

The identity $[\alpha, x^{k-2}, x] = 0$ leads to $\alpha x^{k-1} + x^{k-1}\alpha = x\alpha x^{k-2} + x^{k-2}\alpha x$. Analogously $[\alpha, x^{k-3}, x^2] = 0$ leads to $\alpha x^{k-1} + x^{k-1}\alpha = x^2\alpha x^{k-3} + x^{k-3}\alpha x^2$. We continue in this way and using $[\alpha, x^{l+1}, x^{l-1}] = 0$ we get that $\alpha x^{k-1} + x^{k-1}\alpha = x^{l-1}\alpha x^{l+1} + x^{l+1}\alpha x^{l-1}$. Thus (1) could be written as

$$l(\alpha x^{k-1} + x^{k-1}\alpha) + x^l \alpha x^l = 0. \tag{2}$$

The identity $[\alpha, x^l, x^l] = 0$ gives

$$\alpha x^{k-1} + x^{k-1}\alpha = 2x^l \alpha x^l. \tag{3}$$

Equations (2) and (3) prove the proposition.

Proof of Theorem 3: We'll present the considerations only for the algebra $U18(GR)$. Let $A \in U18(GR)$. We could follow the entries of the powers of A. Let $A^k = B$. Then $b_{11} = a^k = 0$, $b_{1i} = a^{k-1}a_{1i}$ for $i = 2, ..., n-1$, $b_{jn} = a_{jn}a^{k-1}$ for $j = 2, ..., n$. The element b_{1n} is equal to

$$a^{k-1}a_{1n} + a^{k-2}a_{12}a_{2n} + \cdots + a_{12}a_{2n}a^{k-2} + a_{1n}a^{k-1}.$$

Modulo $x^k = 0$ the only nonzero entry of $A^{k+1} = C$ is the entry c_{1n}, equal to

$$\begin{aligned} &a^{k-1}a_{12}a_{2n} + \cdots + a^{k-1}a_{1n-1}a_{n-1n} + a^{k-2}a_{12}a_{2n}a \\ &+ \cdots + aa_{12}a_{2n}a^{k-2} + \cdots + a_{12}a_{2n}a^{k-1}, \end{aligned}$$

which is zero because of Proposition 22.

6. Relation of the above results to algebras with involution

Now we consider finite dimensional Grassmann algebras with involution and some matrix algebras over them.

According to [1] it is enough to consider two involutions for the algebra G - the trivial involution $* = id$, i.e. $id(e_i) = e_i$ for the generators e_i of G and the involution $* = \phi$ acting as $\phi(e_{2k-1}) = e_{2k}$ and $\phi(e_{2k}) = e_{2k-1}$.

Thus for G_5 we could work with the identity involution id only, while for the algebras G'_4 and G'_6 we have two possibilities. We define the skew symmetric elements with respect to the considered involution. Thus we get elements without constant term and we could rely on results already obtained.

Skew symmetric elements of G_5.

Every element of G (in our case of G_5) is ordered in the way used in [13]: First is the element of the field K, then e_1, e_2, e_1e_2, e_3, then we multiply (on the right) by e_3 all previous elements (in the same order), then comes e_4 and all previous elements multiplied by e_4. Thus expressing every element $x \in G_5$ as the vector $x = (a1, a2, ..., a32)$ where ai are the corresponding coefficients, definig "the images" of all summands $e_{i_1}...e_{i_s}$ we form the vector $id(x)$. For example $id(e_2e_3e_5) = id(e_5)id(e_3)id(e_2) = e_5e_3e_2 = -e_2e_3e_5$. Thus the coefficient of $e_2e_3e_5$ in $id(x)$ which is the 23-rd coordinate is $-a23$. The condition $id(x) = -x$ defines the presentation of x, namely

$$\begin{aligned} x_{ss} &= (0,0,0,a4,0,a6,a7,a8,0,a10,a11.a12,a13,a14,a15,0,0,\\ &\quad a18,a19,a20,a21,a22,a23,0,a25,a26,a27,0,a29,0,0,0). \end{aligned}$$

Using a programme written in *Mathematica* [13] we get

Proposition 23 *All skew symmetric with respect to the involution* $* = id$ *elements of* G_5 *are nilpotent of index* ≤ 3.

We denote the set of the elements x_{ss} by (G_5^-, id). Thus we get analogues of Propositions 16–18 and Theorem 1, namely

Corrolary 2 *All matrices of the types described in Propositions* 16–18 *and Theorem* 1 *with entries from* (G_5^-, id) *are nilpotent with index of nilpotency* ≤ 4.

Skew symmetric elements of G_6.

Proposition 13 shows that it is good to consider only the algebra G_6.

We start with the involution id and define the skew symmetric with respect to this involution elements of G_6, denoting them by (G_6^-, id) . Using again the programme from [13] we get one condition on 15 of the 35-th coefficients of the summands of any element x_{ss} of (G_6^-, id) aiming the index of nilpotency of x_{ss} to be ≤ 3.

Next we apply the involution ϕ on the elements x of the algebra G_6. Using the already mentioned above unique order of the summands of x [13] and the condition $\phi(x) = -x$ we get**(denoted by (4))** that

$$\begin{aligned} x_{ss} &= (0,a2,-a2,0,a5,a6,a7,a8,-a5,a7,a6,-a8,0,a14,-a14,0,a17,a18,\\ &\quad a19,a20,a21,a22,a23,a24,a25,a26,a27,a28,a29,a30,a31,a32,-a17, \end{aligned}$$

$$a19, a18, -a20, a25, a27, a26, a28, a21, a23, a22, a24, -a29, a31, a30,$$
$$-a32, 0, a50, -a50, 0, a53, a54, a55, a56, -a53, a55,$$
$$a54, -a56, 0, a62, -a62, 0).$$

We'll explain the form (4) of x_{ss} considering only the 8-th and the 12-th coordinates. The 8-th summand is $a8e_1e_2e_3$. Thus

$$\phi(a8e_1e_2e_3)) = a8\phi(e_3)\phi(e_2)\phi(e_1) = a8e_4e_1e_2 = a8e_1e_2e_4.$$

As $\phi(x) = -x$ we get that the coefficient of $e_1e_2e_4$ which is the coefficient of the 12-th summand of x_{ss} has to be $-a8$.

Using again *Mathematica* [13] we get that $x_{ss}^3 = 0$ leads to the conditions:

$$(a6 + a7)(a2(a21 - a25) - a18a5 + a19a5 + a17a6 - a17a7) = 0;$$
$$(a18 + a19)(a2(-a21 + a25) + a18a5 - a19a5 - a17a6 + a17a7) = 0;$$
$$(a21 + a25)(a2(a21 - a25) - a18a5 + a19a5 + a17a6 - a17a7) = 0.$$

These conditions are on the nine of the 28-th independent coefficients in the presentation (4) of x_{ss}.

The system *Mathematica* could give all classes solutions of the above system which number appears to be 20.

We formulate only one of them.

Let denote by (GS_6^-, ϕ) the set of the skew symmetric with respect to the involution ϕ elements of G_6 which are of the form (4) in which $a7 = -16$, $a19 = -a18$ and $a25 = -a21$. Thus we get

Proposition 24 *All skew symmetric with respect to the involution* $* = \phi$ *elements* x_{ss} *of* (GS_6^-, ϕ) *are nilpotent with index of nilpotency* ≤ 3 *.*

Corrolary 3 *All matrices of the types described in Propositions* 16–18 *and Theorem* 1 *with entries from* (GS_6^-, ϕ) *are nilpotent with index of nilpotency* ≤ 4.

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