STOCHASTIC STABILITY FOR THE STOCHASTIC PERTURBATION OF HAMILTON-POISSON EQUATION IN $\mathbb{R}^3$

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Abstract. For Hamilton-Poisson differential equation in $\mathbb{R}^3$, stochastic perturbations are defined using three-dimensional Wiener process. A Lyapunov function is built for each steady state and it is proved that steady states are stable in probability. Numerical simulations are performed to confirm the new theory presented in this article.

Keywords: stochastic equations, Hamiltonian-Poisson equations, Euler scheme, stochastic Lyapunov function.

2000 *Mathematics Subject Classification*: 26A33, 65L12, 34K50, 35L65, 26A33, 37N99, 60H10, 65C20, 65C30.

1. The Hamilton-Poisson differential equation in $\mathbb{R}^3$

The dynamics of some mechanical systems from technical domain is described by the dynamics of the rigid body with fixed point, or the mathematical pendulum, or oscillators. These mechanical systems are part of the geometric mechanics and belong to a class of differential equations in $\mathbb{R}^3$, with the right side part polynomial functions of degree greater or equal to two.

Classical Hamilton-Poisson differential equations in $\mathbb{R}^3$ are described by the system:

$$\begin{aligned}\dot{x}_1(t) &= \alpha_1 x_2(t) x_3(t),\\ \dot{x}_2(t) &= \alpha_2 x_1(t) x_3(t),\\ \dot{x}_3(t) &= \alpha_3 x_1(t) x_2(t).\end{aligned} \tag{1}$$

For $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = 1$, the system (1) is a system Rabinovich [3].

For $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = -k^2$, $k \in (0, 1)$ the system (1) is a Titeica-Liouville system [5].

For $\alpha_1 = \frac{1}{I_3} - \frac{1}{I_2}$, $\alpha_2 = \frac{1}{I_1} - \frac{1}{I_3}$, $\alpha_3 = \frac{1}{I_2} - \frac{1}{I_1}$, with $I_1 > I_2 > I_3$ the system (1) is the system of rigid body on SO(3) ([7], [8], [9]).

For $\alpha_1 = -\left(\frac{1}{I_2} + \frac{1}{I_3}\right)$, $\alpha_2 = \frac{1}{I_1} + \frac{1}{I_3}$, $\alpha_3 = \frac{1}{I_1} - \frac{1}{I_2}$, with $I_1 > I_2 > I_3$ the system (1) is the system of rigid body on SO(2,1) [2].

The system (1) has the steady states $e_0 = (0, 0, 0)^T$, $e_1 = (m, 0, 0)^T$, $e_2 = (0, m, 0)^T$, $e_3 = (0, 0, m)^T$, with $m \in \mathbb{R}$

Steady state analysis was presented in details in [2], [4], [7], [8], [9].

In reality, stochastic effects can be very important. Recent advances in stochastic differential equations enable us to introduce stochasticity into models describing physical phenomena, as a random noise in the system of differential equation or as environmental fluctuations in parameters.

Let $\{\Omega, \mathcal{F}_t, P\}$ be the probability space with usual notion ([1], [6]), and $(B_1(t), B_2(t), B_3(t))^T = B(t)$ a three-dimensional Winer process. We consider the effect of the environmental fluctuation on the model system (1) and the stochastic stability of the co-existing steady-state associated with the model system. It is assumed that stochastic perturbations of the state variables around their steady-state values in $\mathbb{R}^3$ are Gaussian noise, proportional with the distances between $x = (x_1, x_2, x_2)^T$ and the steady-state e_i, $i = 1, 2, 3$. For this propose it is considered the system (1) with perturbations which are directly proportional to $x_1 - x_{10}$, $x_2 - x_{20}$, respectively $x_3 - x_{30}$, with x_{i0} are the coordinates of e_i, $i = 1, 2, 3$.

The stochastic perturbation of (1) for e_0 is

$$\begin{aligned} dx_1(t) &= (\alpha_1 x_2(t) x_3(t) + a x_1(t))dt + \sigma_1 x_1(t) dB_1(t), \\ dx_2(t) &= (\alpha_2 x_1(t) x_3(t) + b x_2(t))dt + \sigma_2 x_2(t) dB_2(t), \\ dx_3(t) &= (\alpha_3 x_1(t) x_2(t) + c x_3(t))dt + \sigma_3 x_3(t) dB_3(t), \end{aligned} \tag{2}$$

where $a, b, c \in \mathbb{R}$ and $\alpha_i, \sigma_i \in \mathbb{R}$, $i = 1, 2, 3$.

The stochastic perturbation of (1) for e_1 is

$$\begin{aligned} dx_1(t) &= (\alpha_1 x_2(t) x_3(t) + a(x_1(t) - m))dt + \sigma_1 (x_1(t) - m) dB_1(t), \\ dx_2(t) &= (\alpha_2 x_1(t) x_3(t) + b x_2(t))dt + \sigma_2 x_2(t) dB_2(t), \\ dx_3(t) &= (\alpha_3 x_1(t) x_2(t) + c x_3(t))dt + \sigma_3 x_3(t) dB_3(t), \end{aligned} \tag{3}$$

where $a, b, c \in \mathbb{R}$ and $\alpha_i, \sigma_i \in \mathbb{R}$, $i = 1, 2, 3$.

The stochastic perturbation of (1) for e_2 is

$$\begin{aligned} dx_1(t) &= (\alpha_1 x_2(t)x_3(t) + ax_1(t))dt + \sigma_1 x_1(t)dB_1(t), \\ dx_2(t) &= (\alpha_2 x_1(t)x_3(t) + b(x_2(t) - m))dt + \sigma_2(x_2(t) - m)dB_2(t), \\ dx_3(t) &= (\alpha_3 x_1(t)x_2(t) + cx_3(t))dt + \sigma_3 x_3(t)dB_3(t), \end{aligned} \quad (4)$$

where $a, b, c \in \mathbb{R}$ and $\alpha_i, \sigma_i \in \mathbb{R}$, $i = 1, 2, 3$.

The stochastic perturbation of (1) for e_3 is

$$\begin{aligned} dx_1(t) &= (\alpha_1 x_2(t)x_3(t) + ax_1(t))dt + \sigma_1 x_1(t)dB_1(t), \\ dx_2(t) &= (\alpha_2 x_1(t)x_3(t) + bx_2(t))dt + \sigma_2 x_2(t)dB_2(t), \\ dx_3(t) &= (\alpha_3 x_1(t)x_2(t) + c(x_3(t) - m))dt + \sigma_3(x_3(t) - m)dB_3(t), \end{aligned} \quad (5)$$

where $a, b, c \in \mathbb{R}$ and $\alpha_i, \sigma_i \in \mathbb{R}$, $i = 1, 2, 3$.

In [1] a stochastic perturbation of a Hamilton-Poisson system is defined as follows. Let $(\{\cdot, \cdot\}, h, \mathbb{R}^3)$ be a Hamilton-Poisson system of differential equations, given as

$$\dot{x}_i(t) = \{x_i(t), h(x(t))\}, \quad i = 1, 2, 3, \quad (6)$$

and $d_a : \mathbb{R}^3 \to \mathbb{R}$, $a = 1, 2, 3$ are functions of class C^∞. The stochastic perturbation of (6) in the direction d_a, $a = 1, 2, 3$ is the stochastic system

$$dx_i = (\{x_i(t), h(x(t))\} + \sum_{a=1}^{3} \{\{x_i, d_a\}, d_a\}dt + \sum_{a=1}^{3} \{x_i, d_a\}dB_a(t), \; i = 1, 2, 3. \quad (7)$$

Let E be the mean value of the probability space. The stationary solution e_0 of (2) is said to be mean square stable if for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that $E(|x(t)|^2) < \varepsilon$, for any $t \geq 0$. The solution e_0 of (2) is said asymptotically mean square stable if it is mean square stable and $\lim_{t\to\infty} E(|x(t)|^2) = 0$. The stationary solution e_0 of (2) is said to be stable in probability (stochastic) if for any $\varepsilon_1, \varepsilon_2 > 0$, there exists a number $\delta > 0$ such that the solution $x(t)$ satisfies: $P\{|x(t)| > \varepsilon_1\} < \varepsilon_2$, where P denotes the probability of an even.

Theorem 1.[1] *Let open set $D \subset \mathbb{R}^3$, $e_0 \in D$. If there exists a function $V : D \to \mathbb{R}$ such that*

$$\begin{aligned} &k_1 \|x(t)\|^2 \leq V(x(t)) \leq k_2 \|x(t)\|^2, \\ &LV(x(t)) < -k_3 \|x(t)\|^2, \end{aligned} \quad (8)$$

where $k_1, k_2, k_3 \in \mathbb{R}_+$, *and*

$$\begin{aligned} LV(x) &= \alpha_1 x_2 x_3 \frac{\partial V}{\partial x_1} + \alpha_2 x_1 x_3 \frac{\partial V}{\partial x_2} + \alpha_3 x_1 x_2 \frac{\partial V}{\partial x_3} + \\ &+ \frac{1}{2}\sigma_1^2 x_1^2 \frac{\partial^2 V}{\partial x_1^2} + \frac{1}{2}\sigma_2^2 x_2^2 \frac{\partial^2 V}{\partial x_2^2} + \frac{1}{2}\sigma_3^2 x_3^2 \frac{\partial^2 V}{\partial x_3^2} \end{aligned} \quad (9)$$

then the stationary solution e_0 *is stable in probability.*

The function V with the conditions (8), (9) is called the Lyapunov function for e_0.

In Section 2 it is analyzed the stochastic stability for the system (1) according to the α_i, $i = 1, 2, 3$ in steady state e_0. In Section 3 it is analyzed the stochastic stability for the system (1) according to the α_i, $i = 1, 2, 3$ in steady state e_i, $i = 1, 2, 3$. In Section 4, numerical simulation is done and in section 5conclusions and future research directions are presented.

2.The Lyapunov function for e_0

The steady-state analysis is done by building a function on an open set D, $V : D \to \mathbb{R}$, $e_0 \in D$ that satisfies the conditions (8) and (9).

Proposition 2. *If there exist* ω_i, $\omega_i \in \mathbb{R}_+$, $i = 1, 2, 3$, *such that*

$$\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3 = 0 \quad (10)$$

and $a < 0$, $b < 0$, $c < 0$, $|\sigma_1| < \sqrt{2|a|}$, $|\sigma_2| < \sqrt{2|b|}$, $|\sigma_3| < \sqrt{2|c|}$, *then*

$$V(x) = \omega_1 x_1^2 + \omega_2 x_2^2 + \omega_3 x_3^2, \quad (11)$$

satisfies the relations

$$\begin{aligned} &\min_{i=1,2,3}\{\omega_i\}\|x\|^2 \le V(x) \le \max_{i=1,2,3}\{\omega_i\}\|x\|^2, \\ &LV(x) < -\max\{-2a - \sigma_1^2,\ -2b - \sigma_2^2,\ -2c - \sigma_3^2\}\|x\|^2, \end{aligned} \quad (12)$$

where $\|x\|^2 = x_1^2 + x_2^2 + x_3^2$. *It results that the stationary solution* e_0 *is stable in probability.*

Proof. For $V(x)$ given by (11) and (9) results that

$$\begin{aligned} LV(x) &= 2\omega_1 x_1(\alpha_1 x_2 x_3 + a x_1) + 2\omega_2 x_2(\alpha_2 x_1 x_3 + b x_2) \\ &+ 2\omega_3 x_3(\alpha_3 x_1 x_2 + c x_3) \\ &+ \frac{1}{2}(2\sigma_1^2 x_1^2 \omega_1 + 2\sigma_2^2 x_2^2 \omega_2 + 2\sigma_3^2 x_3^2 \omega_3) \\ &= 2(\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3) x_1 x_2 x_3 \\ &+ \omega_1(2a + \sigma_1^2)x_1^2 + \omega_2(2b + \sigma_2^2)x_2^2 + \omega_3(2c + \sigma_3^2)x_3^2. \end{aligned}$$

If $\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3 = 0$ has the solutions $\omega_i > 0$, $i = 1, 2, 3$ and $|\sigma_1| < \sqrt{2|a|}$, $|\sigma_2| < \sqrt{2|b|}$, $|\sigma_3| < \sqrt{2|c|}$, then the relations (12) are true.

From Proposition 2 results that steady-state e_0 is stable in probability.

Corollary 1. *If $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = 1$, a positive solution of the equation $\omega_1 - \omega_2 + \omega_3 = 0$ is $\omega_1 = 1$, $\omega_2 = 2$, $\omega_3 = 1$. The Lyapunov function is given by*

$$V(x) = x_1^2 + 2x_2^2 + x_3^2. \tag{13}$$

Corollary 2. *If $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = -k^2$, a solution of the equation $\omega_1 - \omega_2 - \omega_3 = 0$ is $\omega_1 = 1 + k^2$, $\omega_2 = 1$, $\omega_3 = 1$. The Lyapunov function is given by*

$$V(x) = (1 + k^2)x_1^2 + x_2^2 + x_3^2. \tag{14}$$

Corollary 3. *If $\alpha_1 = \frac{1}{I_3} - \frac{1}{I_2}$, $\alpha_2 = \frac{1}{I_1} - \frac{1}{I_3}$, $\alpha_3 = \frac{1}{I_2} - \frac{1}{I_1}$, $I_1 > I_2 > I_3 > 0$, a positive solution of the equation $\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3 = 0$ is given by $\omega_1 = \frac{1}{\alpha_1}$, $\omega_2 = \frac{2}{|\alpha_2|}$, $\omega_3 = \frac{1}{\alpha_3}$. The Lyapunov function is given by*

$$V(x) = \frac{1}{\alpha_1}x_1^2 + \frac{2}{|\alpha_2|}x_2^2 + \frac{1}{\alpha_3}x_3^2. \tag{15}$$

Corollary 4. *If $\alpha_1 = \frac{1}{I_2} + \frac{1}{I_3}$, $\alpha_2 = -(\frac{1}{I_1} + + \frac{1}{I_3})$, $\alpha_3 = \frac{1}{I_1} - \frac{1}{I_2}$, $I_1 > I_2 > I_3 > 0$, a positive solution of the equation $\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3 = 0$ is of the form $\omega_1 = \frac{1}{|\alpha_1|}$, $\omega_2 = \frac{2}{\alpha_2}$, $\omega_3 = \frac{1}{|\alpha_3|}$ and the Lyapunov function is given by*

$$V(x) = \frac{1}{|\alpha_1|}x_1^2 + \frac{2}{\alpha_2}x_2^2 + \frac{1}{|\alpha_3|}x_3^2. \tag{16}$$

The Poisson structure that defines the system of differential equations of free rigid body motion in SO(3) is

$$\{x_1, x_2\} = -x_2, \qquad \{x_1, x_3\} = x_2, \qquad \{x_2, x_3\} = -x_1. \tag{17}$$

The stochastic perturbation of the system of differential equation, corresponding to the rigid body in SO(3) along the directions

$$d_1(x) = a_1 x_1(t), \qquad d_2(x) = a_2 x_2(t), \qquad d_3(x) = a_3 x_3(t), \tag{18}$$

where $a_i \in \mathbb{R}$, $i = 1, 2, 3$ is

$$\begin{aligned} dx_1(t) &= (\alpha_1 x_2(t) x_3(t) - (a_2^2 + a_3^2) x_1(t)) dt - a_2 x_3(t) dB_2(t) + a_3 x_2(t) dB_3(t), \\ dx_2(t) &= (\alpha_2 x_1(t) x_3(t) - (a_1^2 - a_3^2) x_2(t)) dt - a_3 x_{33}(t) dB_1(t) - a_3 x_1(t) dB_3(t), \\ dx_3(t) &= (\alpha_3 x_1(t) x_2(t) - (a_1^2 + a_2^2) x_3(t)) dt - a_1 x_2(t) dB_1(t) + a_2 x_1(t) dB_2(t). \end{aligned} \tag{19}$$

Stochastic equations are obtained from (17), (18) and (7).

Proposition 3. *The steady state e_0 is stable in probability, for all $\alpha_i \in \mathbb{R}$, $i = 1, 2, 3$.*

Proof. For $\omega_1 = 1$, $\omega_2 = 1$, $\omega_3 = 1$, and $\alpha_1 = \frac{1}{I_3} - \frac{1}{I_2}$, $\alpha_2 = \frac{1}{I_1} - \frac{1}{I_3}$, $\alpha_3 = \frac{1}{I_2} - \frac{1}{I_1}$ it results that $\omega_1 \alpha_1 + \omega_2 \alpha_2 + \omega_3 \alpha_3 = 0$. The Lyapunov function is then given by

$$V(x) = x_1^2 + x_2^2 + x_3^2. \tag{20}$$

From (20), and (9) it results that

$$LV(x) = -(a_2^2 + a_3^2) x_1^2 - (a_3^2 + a_1^2) x_2^2 - (a_1^2 + a_2^2) x_3^2 \leq - \max(a_2^2 + a_3^2, a_3^2 + a_1^2, a_1^2 + a_2^2) \|x\|^2. \tag{21}$$

3.The Lyapunov function for e_i, $i = 1, 2, 3$

Let us consider the study for the steady-state $e_1 = (m, 0, 0)^T$.

Proposition 4. *If there exist ω_i, $i = 1, 2, 3$, $\omega_i \in \mathbb{R}_+$ so that*

$$\alpha_1 \omega_1 + \alpha_2 \omega_2 + \alpha_3 \omega_3 = 0, \tag{22}$$

$a < 0$, $b < 0$, $c < 0$, $|\sigma_1| < \sqrt{2|a|}$, $|\sigma_2| < \sqrt{2|b|}$, $|\sigma_3| < \sqrt{2|c|}$, $m \in \mathbb{R}$ and

$$|m\alpha_1| < \frac{\sqrt{(2b + \sigma_2^2)(2c + \sigma_3^2)\omega_2 \omega_3}}{\omega_1}, \tag{23}$$

then the Lyapunov function has the form

$$V_1(x) = V_1(x_1 - m, x_2, x_3) = \omega_1(x^1 - m)^2 + \omega_2 x_2^2 + \omega_3 x_3^2. \qquad (24)$$

Proof. Let $V_1(x)$ be given by (24). From (3) and (2) it results that

$$\begin{aligned} LV_1(x) &= 2x_1x_2x_3(\omega_1\alpha_1 + \omega_2\alpha_2 + \omega_3\alpha_3) + \omega_1(x_1 - m)^2(2a + \sigma_1^2) \\ &+ \omega_2 x_2^2(2b + \sigma_2^2) + \omega_3 x_3^2(2c + \sigma_3^2) - 2\omega_1\alpha_1 m x_2 x_3. \end{aligned}$$

From (22) and (23) we have the following inequations

$$\min\{\omega_1, \omega_2, \omega_3\}((x_1 - m)^2 + x_2^2 + x_3^2) \leq V_1(x) \leq \max\{\omega_1, \omega_2, \omega_3\}((x_1 - m)^2 + x_2^2 + x_3^2),$$

$$\begin{aligned} V_1(x) &= \omega_1(x_1 - m)^2(2a + \sigma_1^2) + \omega_2 x_2^2(2b + \sigma_2^2) + \omega_3 x_3^2(2c + \sigma_3^2) - 2\omega_1\alpha_1 m x_2 x_3 \\ &\leq -\max(-\omega_1(2a + \sigma_1^2), -\omega_2(2b + \sigma_2^2), -\omega_3(2c + \sigma_3^2))((x_1 - m)^2 + x_2^2 + x_3^2). \end{aligned}$$

Thus $V_1(x)$ is Lyapunov function for e_1 and the steady-state e_1 is stable in probability.

The result for the steady-state $e_2 = (0, m, 0)^T$ is the following proposition.

Proposition 5. *If there exist ω_i, $i = 1, 2, 3$, $\omega_i \in \mathbb{R}_+$ such that*

$$\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3 = 0, \qquad (25)$$

$a < 0$, $b < 0$, $c < 0$, $|\sigma_1| < \sqrt{2|a|}$, $|\sigma_2| < \sqrt{2|b|}$, $|\sigma_3| < \sqrt{2|c|}$, $m \in \mathbb{R}$ *and*

$$|m\alpha_2| < \frac{\sqrt{(2a + \sigma_1^2)(2c + \sigma_3^2)\omega_1\omega_3}}{\omega_2}, \qquad (26)$$

the Lyapunov function is given by

$$V_2(x) = \omega_1 x_1^2 + \omega_2(x_2 - m)^2 + \omega_3 x_3^2. \qquad (27)$$

The proof is similar to that for the Proposition 4.

Let us consider the steady-state $e_3 = (0, 0, m)^T$.

Proposition 6. *If there exist* ω_i, $i = 1, 2, 3$, $\omega_i \in \mathbb{R}_+$ *so that*

$$\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3 = 0, \tag{28}$$

$a < 0$, $b < 0$, $c < 0$, $|\sigma_1| < \sqrt{2|a|}$, $|\sigma_2| < \sqrt{2|b|}$, $|\sigma_3| < \sqrt{2|c|}$, $m \in \mathbb{R}$ *and*

$$|m\alpha_3| < \frac{\sqrt{(2a + \sigma_1^2)(2b + \sigma_2^2)\omega_1\omega_2}}{\omega_3}, \tag{29}$$

then the Lyapunov function is

$$V_3(x) = \omega_1 x_1^2 + \omega_2 x_2^2 + \omega_3(x_3 - m)^2. \tag{30}$$

Using the Propositions 4, 5, 6 the following results are obtained.

Corollary 5. *If* $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = 1$, *then* $\omega_1 = 1$, $\omega_2 = 2$, $\omega_3 = 3$

a. *If*

$$|m| < \sqrt{2(2b + \sigma_2^2)(2c + \sigma_3^2)}, \tag{31}$$

the steady-state e_1 *is stable in probability.*

b. *If*

$$|m| < \frac{\sqrt{(2a + \sigma_1^2)(2c + \sigma_3^2)}}{2} \tag{32}$$

the steady-state e_2 *is stable in probability.*

c. *If*

$$|m| < \sqrt{2(2a + \sigma_1^2)(2b + \sigma_2^2)}, \tag{33}$$

the steady-state e_3 *is stable in probability.*

Corollary 6. *If* $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = -k^2$, $k \in (0, 1)$, *then* $\omega_1 = 1 + k^2$, $\omega_2 = 1$, $\omega_3 = 1$.

a. *If*

$$|m| < \frac{\sqrt{(2b + \sigma_2^2)(2c + \sigma_3^2)}}{1 + k^2}, \tag{34}$$

the steady-state e_1 *is stable in probability.*

b. *If*

$$|m| < \sqrt{(2a+\sigma_1^2)(2c+\sigma_3^2)(1+k^2)}, \tag{35}$$

the steady-state e_2 is stable in probability.

c. *If*

$$|m| < \sqrt{2(2a+\sigma_1^2)(2b+\sigma_2^2)(1+k^2)}, \tag{36}$$

the steady-state e_3 is stable in probability.

Corollary 7. *If $\alpha_1 = \frac{1}{I_3} - \frac{1}{I_1}$, $\alpha_2 = \frac{1}{I_1} - \frac{1}{I_2}$, $\alpha_3 = \frac{1}{I_2} - \frac{1}{I_3}$, $I_1 > I_2 > I_3 > 0$, then $\omega_1 = \frac{1}{\alpha_1}$, $\omega_2 = \frac{1}{|\alpha_2|}$, $\omega_3 = \frac{1}{\alpha_3}$.*

a. *If*

$$|m| < \sqrt{\frac{2(2b+\sigma_2^2)(2c+\sigma_3^2)}{\alpha_3|\alpha_2|}}, \tag{37}$$

the steady-state e_1 is stable in probability.

b. *If*

$$|m| < \frac{1}{2}\sqrt{\frac{(2a+\sigma_1^2)(2c+\sigma_3^2)}{\alpha_1\alpha_3}}, \tag{38}$$

the steady-state e_2 is stable in probability.

c. *If*

$$|m| < \sqrt{\frac{2(2a+\sigma_1^2)(2b+\sigma_2^2)}{\alpha_1|\alpha_2|}}, \tag{39}$$

the steady-state e_3 is stable in probability.

Corollary 8. *If $\alpha_1 = -(\frac{1}{I_2} + \frac{1}{I_3})$, $\alpha_2 = \frac{1}{I_1} + \frac{1}{I_2}$, $\alpha_3 = \frac{1}{I_1} - \frac{1}{I_2}$, $I_1 > I_2 > I_3 > 0$, Then $\omega_1 = \frac{1}{|\alpha_1|}$, $\omega_2 = \frac{1}{\alpha_2}$, $\omega_3 = \frac{1}{|\alpha_3|}$.*

a. *If*

$$|m| < \sqrt{\frac{2(2b+\sigma_2^2)(2c+\sigma_3^2)}{\alpha_2|\alpha_3|}}, \tag{40}$$

the steady-state e_1 is stable in probability.

b. *If*

$$|m| < \frac{1}{2}\sqrt{\frac{(2a+\sigma_1^2)(2c+\sigma_3^2)}{|\alpha_1|\alpha_3}}, \tag{41}$$

the steady-state e_2 *is stable in probability.*

c. *If*

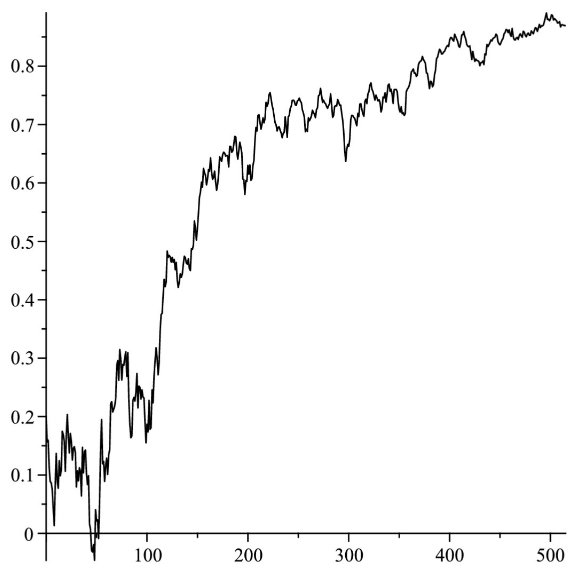
$$|m| < \sqrt{\frac{2(2a+\sigma_1^2)(2b+\sigma_2^2)}{|\alpha_1|\alpha_2}}, \tag{42}$$

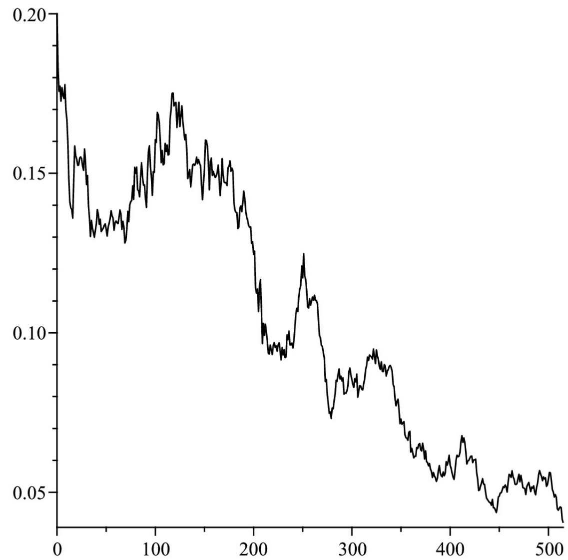
the steady-state e_3 *is stable in probability.*

4. Numerical simulation

Numerical simulation can be done using Matlab or Maple 12, using the Euler stochastic method. Simulation results confirm the exposed theory.

For $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = -k^2$, $k =$ and m satisfying (32), orbits $(i, x_1(i,\omega))$, $(i, x_2(i,\omega))$, $(i, x_2(i,\omega))$ are obtained. Their graphical representation are shown in Figure 1, Figure 2, Figure 3.

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0.15
0.10
0.05
0
100
200
300
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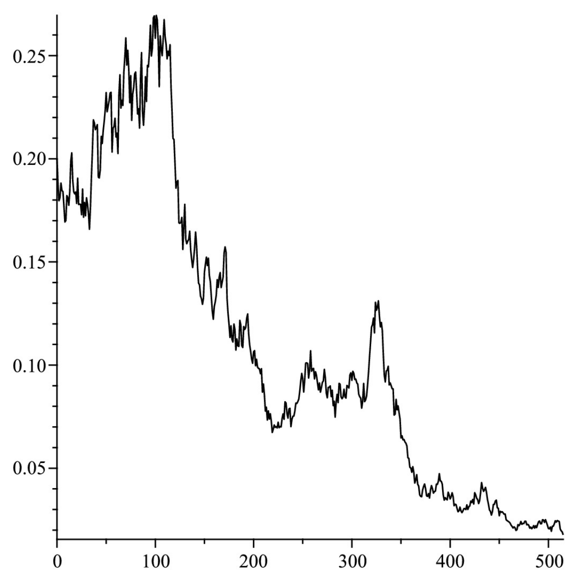
500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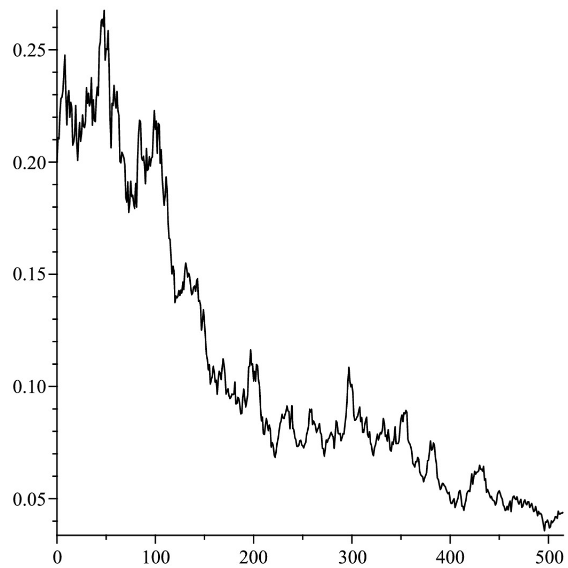
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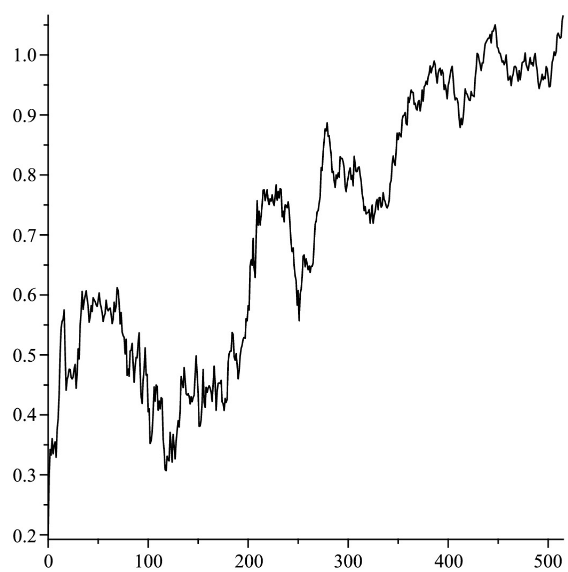
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0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
100
200
300
400
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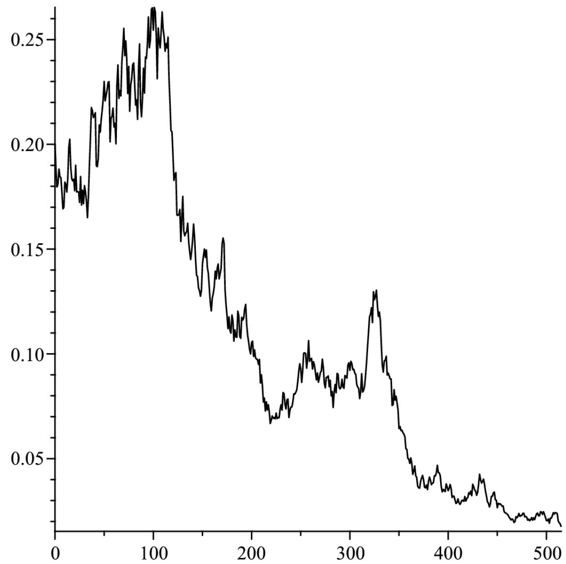
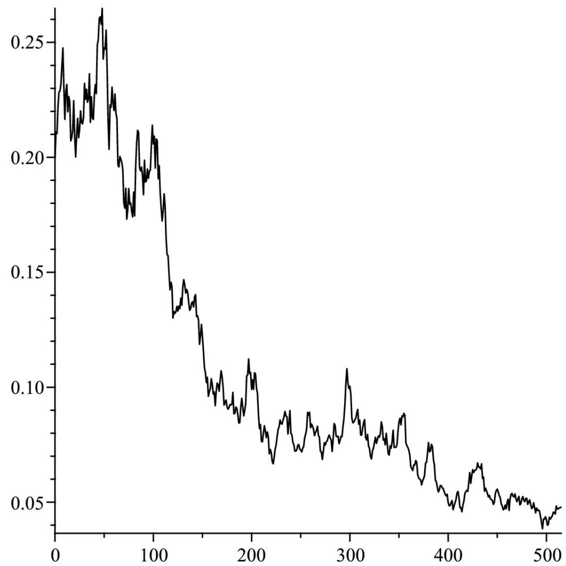


For m satisfying (33), we obtain Figure 4, Figure 5, Figure 6.
For m satisfying (34), we obtain Figure 7, Figure 8, Figure 9.
Similarly simulations are performed for all the cases described in Corollary 1, 2, 3, 4, 5, 7, 8.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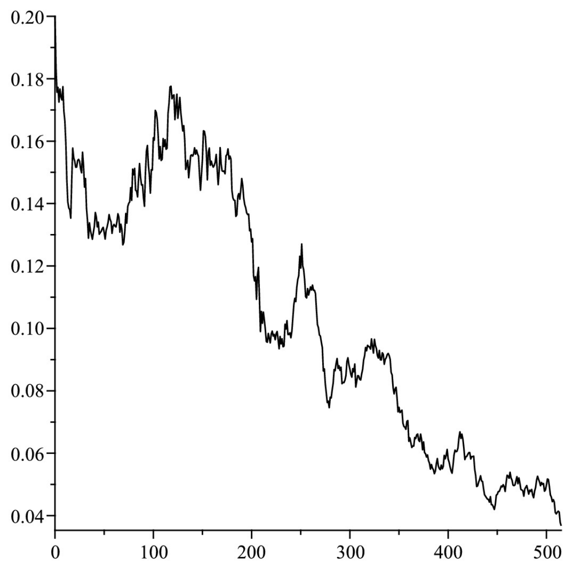
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0.08
0.06
0.04
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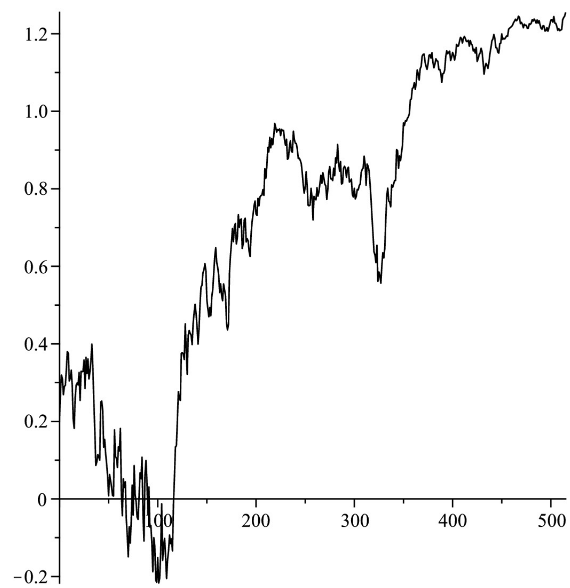

5. Conclusions

In this paper, for classical Hamilton-Poisson systems of $\mathbb{R}^3$ (Rabinovicz, Titeica-Liouville, rigid body in SO(3), rigid body in SO(2,1)) with linear control, stochastic perturbation was defined, associated to the steady states e_0, e_i, $i = 1, 2, 3$. For stochastic differential equations system, Lyapunov function was determined, and also the values of the parameters that describe the system, such that the steady states to be stable in probability. The method used in this paper can be applied to other Hamilton-Poisson systems from $\mathbb{R}^3$ such as the Rikitake [11] and planar motions of an autonomous underwater vehicle [10].

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