HAAR AND LEGENDRE WAVELETS COLLOCATION METHODS FOR THE NUMERICAL SOLUTION OF SCHRODINGER AND WAVE EQUATIONS

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ABSTRACT. Based on collocation with Haar and Legendre wavelets, two efficient methods are being proposed for the numerical solution of linear and nonlinear differential equations. The present method is developed in two stages. In the initial stage, it is developed for Haar wavelets. In order to obtain higher accuracy, Haar wavelets are replaced by Legendre wavelets at the second stage. A comparative analysis of the performance of Legendre wavelets collocation and quintic B-spline collocation method is carried out. The analysis indicates that there is a higher accuracy obtained by Legendre wavelets decomposition, which is in the form of a multi-resolution analysis of the function. Through this analysis the solution is found on the coarse grid points and then refined towards higher accuracy by increasing the level of the wavelets. A distinct feature of the proposed methods is their simple applicability for a variety of boundary conditions. Numerical examples show better accuracy of the proposed method based on Legendre wavelets for a variety of benchmark problems.

1. INTRODUCTION

Nonlinear wave equations appear in various areas of Physics, Engineering, Biological Sciences, Geological Sciences and many other places. Recently many new approaches to nonlinear wave equations have been proposed, for example, tanh-function method [25, 27], Jocobian elliptic function expansion method [26, 28], F-expansion method [35], variational iteration method [29, 30], Adomian method [31, 32], variational approache [36], and homotopy perturbation method [33, 34] and so on.

In the recent years the wavelet approach is becoming more popular in the field of numerical approximations. Different types of wavelets and approximating functions have been used for this purpose. The examples include Daubechies [13], Battle-Lemarie [14], B-spline [11], Chebyshev [15], Legendre [16, 17] and Haar wavelets [10, 12, 18, 19]. On account of their simplicity, Haar wavelets have received the attention of many researchers. A short introduction to the Haar wavelets and its applications can be found in [18, 21, 22, 23, 24]. Legendre wavelets, which are another type of wavelets, use Legendre polynomials as their basis functions. They have good interpolating properties and give better accuracy for smaller number of collocation points. Applications of Legendre wavelets for numerical approximations can be found in the references [4, 5, 20].

The model equation which describes the light wave envelope is given by the well khown nonlinear Schrodinger equation (NLSE). Let us consider a higher order NLSE in the form

$$iE_z = \frac{\beta_2}{2}E_{tt} + \frac{\beta_4}{24}E_{tttt} - \gamma_1|E|^2E - \gamma_2|E|^4E + i\alpha_1(|E|^2E)_t, \tag{1}$$

where E(z,t) is the slowly varying envelope of the electric field, the subscripts z and t are the spatial and temporal partial derivatives in retard time coordinates, and β_2 and η_4 represent the group velocity dispersion (GVD), and fourth-order dispersion (FOD), respectively. γ_1 and γ_2 are the cubic and quintic nonlinearity coefficients, respectively. α_1 represents the self-steepening effect coefficient. When $\beta_4 = \gamma_2 = \alpha_1 = 0$, Eq. (1) reduces to the standard NLSE, which describes the propagation of picosecond pulses in optical fibers.

The objective of this research is to construct a simple collocation method with the Haar and Legendre basis functions for the numerical solution of wave equation with initial condition and NLSE with boundary conditions. To test applicability of the Haar and Legendre wavelets, we apply proposed method for several examples.

2. HAAR WAVELET

The one dimensional Haar wavelet family for $x \in [0,1)$ is defined as

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\alpha, \beta), \\ -1 & \text{for } x \in [\beta, \gamma), \\ 0 & \text{elsewhere,} \end{cases} \tag{2}$$

where

$$\alpha = \frac{k}{m}, \quad \beta = \frac{k+0.5}{m}, \quad \gamma = \frac{k+1}{m}, \tag{3}$$

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with $m = 2^j$, $j = 0, 1, ..., J$, $M = 2^J$ and $k = 0, 1, ..., m - 1$. The integer j indicates level of the wavelet and k is translation parameter. The relation between i, m and k is expressed as $i = m + k + 1$. In the case with minimal values $m = 1, \quad k = 0$, we have $i = 2$. The maximal value of i is $i = 2M = 2^{J+1}$.

For $i = 1$, the function $h_1(x)$ is the scaling function for the family of Haar wavelet is defined as

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [0, 1), \\ 0 & \text{elsewhere.} \end{cases} \tag{4}$$

Any square integrable function $f(x)$ in the interval $(0, 1)$ can be expressed as an infinite sum of Haar wavelet as

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x), \tag{5}$$

where $a_i, i = 1, 2, ...$ are the Haar coefficients. The above series terminates as finite terms if $f(x)$ is piecewise constant or it can be approximated as piecewise constant during each subinterval.

The following notations are introduced:

$$p_{i,1}(x) = \int_0^x h_i(x')dx', \tag{6}$$

$$p_{i,\nu+1}(x) = \int_0^x p_{i,\nu}(x')dx', \quad \nu = 1, 2, ... \tag{7}$$

$$c_{i,1}(x) = \int_0^1 p_{i,1}(x')dx', \tag{8}$$

These integrals can be evaluated using Eq. (2) and are expressed as follows:

$$p_{i,1}(x) = \begin{cases} x - \alpha & \text{for } x \in [\alpha, \beta), \\ \gamma - x & \text{for } x \in [\beta, \gamma), \\ 0 & \text{elsewhere,} \end{cases} \tag{9}$$

$$p_{i,2}(x) = \begin{cases} \frac{1}{2}(x - \alpha)^2 & \text{for } x \in [\alpha, \beta), \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - x)^2 & \text{for } x \in [\beta, \gamma), \\ \frac{1}{4m^2} & \text{for } x \in [\gamma, 1), \\ 0 & \text{elsewhere.} \end{cases} \tag{10}$$

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3. LEGENDRE WAVELET

For any positive integer k, the Legendre wavelets family is defined [4] as given below:

$$\psi_{m,n}(x) = \begin{cases} \sqrt{m + \frac{1}{2}} 2^{\frac{k}{2}} L_m(2^k x - 2n + 1) & \text{for } x \in \left[\frac{2n-2}{2^k}, \frac{2n}{2^k}\right), \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

where $n = 1, 2, ..., 2^{k-1}$ and $m = 0, 1, 2, ...$. Here, $L_m(x)$ are the Legendre polynomials of order m which are defined on the interval $[-1, 1]$. Legendre polynomials can be calculated recursively with the help of following relations:

$$L_0(x) = 1, \quad L_1(x) = x, \tag{12}$$

$$L_{k+1}(x) = \left(\frac{2k+1}{k+1}\right) x L_k(x) - \left(\frac{k}{k+1}\right) L_{k-1}(x), \quad k = 1, 2, 3, ... \tag{13}$$

Equivalently, for any positive integer k, we can define the Legendre wavelets family as:

$$\psi_i(x) = \begin{cases} \sqrt{m + \frac{1}{2}} 2^{\frac{k}{2}} L_m(2^k x - 2n + 1) & \text{for } x \in \left[\frac{2n-2}{2^k}, \frac{2n}{2^k}\right), \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

where $n = 1, 2, ..., 2^{k-1}$, $m = 0, 1, 2, ...$ and $i = n + 2^{k-1} m$. Any function $f(x)$ which is square integrable in the interval $(0, 1)$ can be expanded by Legendre wavelets series [5] as

$$f(x) = \sum_{m=0}^{\infty} \sum_{n=1}^{2^{k-1}} a_{mn} \psi_{mn}(x) = \sum_{i=1}^{\infty} a_i \psi_i(x). \tag{15}$$

For approximations, the above series may be truncated and written as follows:

$$f(x) = \sum_{i=1}^{N} a_i \psi_i, \tag{16}$$

where $N = 2^{k-1} M$ and Legendre polynomials used in the approximation are of degree less than M. The following notations are introduced.

$$\psi_i^1(x) = \int_0^x \psi_i(x') dx', \tag{17}$$

$$\psi_i^{\nu+1}(x) = \int_0^x \psi_i^{\nu}(x') dx', \quad \nu = 1, 2, ... \tag{18}$$

$$c_i = \int_0^1 \psi_i^1(x') dx'. \tag{19}$$

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The best way to understand the wavelets is through a multi-resolution analysis. Given a function $f \in L_2(\mathbb{R})$ a multi-resolution analysis (MRA) of $L_2(\mathbb{R})$ produces a sequence of subspaces $V_j, V_{j+1}, ...$ such that the projections of f onto these spaces give finer and finer approximations of the function f as $j \to \infty$.

Definition 1. *(Multi-resolution analysis). A multi-resolution analysis of $L_2(\mathbb{R})$ is defined as a sequence of closed subspaces $V_j \subset L_2(\mathbb{R})$, $j \in \mathbb{Z}$ with the following properties*
(i)... $\subset V_{-1} \subset V_0 \subset V_1 \subset ...$
(ii) The spaces V_j satisfy $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L_2(\mathbb{R})$ and $\bigcup_{j \in \mathbb{Z}} V_j = 0$.
(iii) If $f(x) \in V_0, f(2^j x) \in V_j$, i.e. the spaces V_j are scaled versions of the central space V_0.
(iv) If $f(x) \in V_0, f(2^j x - k) \in V_j$, i.e. all the V_j are invariant under translation.
(v) There exists $\Phi \in V_0$ such that $\Phi(x - k); k \in \mathbb{Z}$ is a Riesz basis in V_0.

The space V_j is used to approximate general functions by defining appropriate projection of these functions onto these spaces. Since the union of all the V_j is dense in $L_2(\mathbb{R})$, so it guarantees that any function in $L_2(\mathbb{R})$ can be approximated arbitrarily close by such projections. As an example, the space V_j can be defined like

$$V_j = W_{j-1} \oplus V_{j-1} = W_{j-1} \oplus W_{j-2} \oplus V_{j-2} = ... = \bigoplus_{j=1}^{J+1} W_j \oplus V_0,$$

then the scaling function $h_1(x)$ generates an MRA for the sequence of spaces $\{V_j, j \in \mathbb{Z}\}$ by translation and dilation as defined in Eqs. (2) and (4). For each j the space W_j serves as the orthogonal complement of V_j in V_{j+1}. The space W_j include all the functions in V_{j+1} that are orthogonal to all those in V_j under some chosen inner product. The set of functions which from basis for the space W_j are called wavelets [8, 9].

4. Haar wavelet collocation method (HWCM)

To construct a simple and accurate HWCM for the second-order boundary value problems

$$f'' = \phi(x, f, f'), \tag{20}$$

with boundary conditions

$$f(0) = \alpha, \quad f(1) = \beta, \tag{21}$$

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the wavelet approximations for the highest derivatives of f are given by

$$f''(x) = \sum_{i=1}^{2M} a_i h_i(x). \tag{22}$$

The following collocation points are considered:

$$x_j = \frac{j - 0.5}{2M}, \quad j = 1, 2, ..., 2M. \tag{23}$$

By double integrating of $f''(x)$ we have:

$$f'(x) = \int_0^1 \int_0^x f''(x) dx = \sum_{i=1}^{2M} a_i \int_0^1 \int_0^x h_i(x) dx. \tag{24}$$

Then

$$f'(x) = \beta - \alpha + \sum_{i=1}^{2M} a_i (p_{i,1}(x) - c_{i,1}). \tag{25}$$

The value of $f(x)$ can be expressed as

$$f(x) = \alpha + (\beta - \alpha)x + \sum_{i=1}^{2M} a_i (p_{i,2}(x) - x c_{i,1}). \tag{26}$$

5. Legendre wavelet collocation method (LWCM)

To construct better approximation for the second-order boundary value problems

$$f'' = \phi(x, f, f'), \tag{27}$$

with boundary conditions

$$f(0) = \alpha, \quad f(1) = \beta, \tag{28}$$

we switch from Haar wavelet to Legendre wavelets basis. The wavelet approximations for the highest derivatives of f are given by

$$f''(x) = \sum_{i=1}^{N} a_i \psi_i(x). \tag{29}$$

The following collocation points are considered:

$$x_i = \frac{i - 0.5}{N}, \quad i = 1, 2, ..., N. \tag{30}$$

By double integrating of $f''(x)$ we have:

$$f'(x) = \int_0^1 \int_0^x f''(x)dx = \sum_{i=1}^N a_i \int_0^1 \int_0^x \psi_i(x)dx. \tag{31}$$

Then

$$f'(x) = \beta - \alpha + \sum_{i=1}^N a_i(\psi_i^1(x) - c_i). \tag{32}$$

The value of $f(x)$ can be expressed as

$$f(x) = \alpha + (\beta - \alpha)x + \sum_{i=1}^N a_i(\psi_i^2(x) - xc_i). \tag{33}$$

6. Error analysis

In this section we describe the error analysis for proposed methods.

6.1. Haar wavelets

Lemma 1. *Assume that $f(x) \in L_2(\mathbb{R})$ with the bounded first derivative on $(0,1)$, then the error norm at Jth level satisfies the following inequality*

$$\| e_J(x) \|_2 \le G2^{-(3)2^{J-1}}, \tag{34}$$

where $e_J(x) = f(x) - f_J(x)$ and $G = C\sqrt{\frac{k}{7}}$, is some real constant.

Proof. For proof see [2].

6.2. Legendre wavelets

Lemma 2. *Suppose that the function $f(x)$ is piecewise constant or may be approximated as piecewise constant, then we can approximates $f(x)$ as*

$$f(x) \approx \sum_{i=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{i,m} \psi_{m,i}(x) = f_M(x), \tag{35}$$

then $f_M(x)$ approximate $f(x)$ with the following error norm

$$\| f(x) - f_M(x) \|_2 \le \frac{1}{M!} \frac{1}{2^{Mk}} sup_{x \in [0,1]} |f^{(M)}(x)|. \tag{36}$$

Proof. For proof see [1].

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7. Numerical experiments

In this section, numerical results of six numerical experiments are presented in order to demonstrate the accuracy of the proposed methods. Performance of the present methods is compared with the existing method in literature [6].

The notation L_∞ will be used for the maximum absolute errors;

$$L_\infty = Max|f_j^e - f_j^a|, \tag{37}$$

where f_j^e and f_j^a are the exact and approximate solution respectively at the jth collocation point x_j, such that for the HWCM we have $j = 1, ..., 2M$ and for the LWCM we have $j = 1, ..., N$.

Example 1. *Consider the general nonlinear wave PDE*

$$y_t = G(y, y_x, y_{xx}, ...) = 0. \tag{38}$$

In [7] by applying the tanh-coth method, the Eq. (38) is reduced to an ordinary differential equation (ODE) given by

$$y' = \alpha + \beta y + \gamma y^2, \tag{39}$$

where α, β and γ are constants.
If we take $\alpha = 1$, $\beta = 0$ and $\gamma = -1$, we have

$$y' = 1 - y^2, \tag{40}$$

with boundary condition $y(0) = 0$. The exact solution is given by

$$y(x) = tanh(x). \tag{41}$$

HWCM and LWCM are applied to this problem and L_∞ for different values of J and N are shown in Table 1. From Table 1 it is clear that LWCM performs much better than HWCM.

Example 2. *If we take in Example 1, $\alpha = 1$, $\beta = 0$ and $\gamma = -4$, we have*

$$y' = 1 - 4y^2, \tag{42}$$

with boundary condition $y(0) = 0$.
The exact solution is given by

$$y(x) = \frac{tanh(x)}{1 + (tanh(x))^2}. \tag{43}$$

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HWCM and LWCM are applied to this problem and L_∞ for different values of J and N are shown in Table 2. The numerical results show that better accuracy can be achieved for LWCM by increasing the number of collocation points.

Example 3. *If we take in Example 1, $\alpha = 1$, $\beta = -2$ and $\gamma = 2$, we have*

$$y' = 1 - 2y + 2y^2, \tag{44}$$

with boundary condition $y(0) = 0$.
The exact solution is given by

$$y(x) = \frac{tan(x)}{1 + tan(x)}. \tag{45}$$

HWCM and LWCM are applied to this problem and L_∞ for different values of J and N are shown in Table 3. From Table 3 it is clear that LWCM performs much better than HWCM.

Example 3. *We solve*

$$y''(x) + (x^2 - 6x - 1)y'(x) + (5x - x^2 + 6)y(x) = e^x - x^2 + 5x + 6, \tag{46}$$

over $[0, 1]$ with boundary conditions $y(0) = 1$, $y(1) = 1 + e$.
Where the analytic solution is

$$y(x) = xe^x + 1. \tag{47}$$

The LWCM is applied to this problem and L_∞ for different values of N are shown in Table 4. It can be seen from Table 4 that the accuracy of the LWCM by increasing the level of resolution N is increases. The same problem is solved in [6] by using quintic B-spline collocation method and the maximum absolute errors recorded there in are 4.498×10^{-11} for the N=80 collocation points whereas the maximum absolute errors of our algorithm as listed in Table 4 are 1.415×10^{-17} for the N=32 collocation points.

Table 1
Comparison of HWCM and LWCM in term of L_∞ for Example 1.

J	2M	HWCM	N	LWCM
1	4	3.002×10^{-3}	4	1.134×10^{-4}
2	8	7.801×10^{-4}	8	1.046×10^{-6}
3	16	1.971×10^{-4}	16	2.010×10^{-12}
4	32	4.941×10^{-5}	32	1.304×10^{-16}
5	64	1.236×10^{-5}	64	1.552×10^{-19}

Table 2

Comparison of HWCM and LWCM in term of L_∞ for Example 2.

J	2M	HWCM	N	LWCM
1	4	5.273×10^{-3}	4	3.568×10^{-3}
2	8	1.501×10^{-3}	8	1.101×10^{-5}
3	16	3.901×10^{-4}	16	3.318×10^{-8}
4	32	9.855×10^{-5}	32	2.033×10^{-12}
5	64	2.470×10^{-5}	64	6.539×10^{-17}

Table 3

Comparison of HWCM and LWCM in term of L_∞ for Example 3.

J	2M	HWCM	N	LWCM
1	4	9.544×10^{-3}	4	1.155×10^{-3}
2	8	2.993×10^{-3}	8	7.433×10^{-6}
3	16	8.498×10^{-4}	16	3.733×10^{-10}
4	32	2.274×10^{-4}	32	8.733×10^{-14}
5	64	5.888×10^{-5}	64	6.520×10^{-18}

Example 4. *We consider*

$$y''(x) = \frac{-1}{2} y(x) y'(x), \qquad x \in [0,1], \tag{48}$$

with the boundary conditions $y(0) = -\frac{4}{5}, y(1) = -1$. Where the analytic solution is $y(x) = \frac{4}{x-5}$.
Maximum absolute errors for different values of N and J are shown in Table 5. The same problem is solved in [6] by using quintic B-spline collocation method and the maximum absolute errors recorded there in are 1.125×10^{-6} for the N=80 collocation points whereas the maximum absolute errors of our algorithm as listed in Tabel 5 are 2.033×10^{-10} for the N=32 collocation points.

Table 4

Comparison of LWCM and method presented in [6] in term of L_∞ for Example 4.

N	LWCM	n	Method presented in [6]
4	7.805×10^{-5}	5	2.885×10^{-6}
8	1.002×10^{-10}	10	1.835×10^{-7}
16	1.256×10^{-11}	20	1.151×10^{-8}
20	4.499×10^{-15}	40	7.202×10^{-10}
32	6.675×10^{-19}	80	4.498×10^{-11}

Table 5
Comparison of LWCM, HWCM and method presented in [6] in term of L_∞ for Example 5.

N	LWCM	J	HWCM	n	Method presented in [6]
4	1.145×10^{-5}	1	2.391×10^{-5}	10	3.663×10^{-3}
8	3.895×10^{-8}	2	6.422×10^{-6}	20	2.642×10^{-4}
16	2.670×10^{-9}	3	1.619×10^{-6}	40	1.777×10^{-5}
32	2.033×10^{-10}	4	4.072×10^{-7}	80	1.125×10^{-6}

Example 5. *Consider a standard nonlinear Schrodinger equation (NLSE)*

$$iE_z = \frac{\beta_2}{2}E_{tt} - \gamma_1 |E|^2 E. \tag{49}$$

In [3] the nonlinear sub-ODE method for Eq. (49) is of the form

$$y''(x) = y(x) + 3y(x)^2, \tag{50}$$

with boundary conditions $y(0) = -\frac{1}{2}$, $y(1) = -0.39322$.
The exact solution is given by

$$y(x) = \frac{-1}{1 + cosh(x)}. \tag{51}$$

Maximum absolute errors for different values of N and J are shown in Table 6.

Table 6
Comparison of HWCM and LWCM in term of L_∞ for Example 6.

N	M	k	LWCM	J	HWCM
4	2	2	7.621×10^{-5}	1	9.607×10^{-5}
8	4	2	3.358×10^{-7}	2	2.830×10^{-5}
16	8	2	1.667×10^{-10}	3	8.904×10^{-6}

8. Conclusion

Two efficient methods LWCM and HWCM have been proposed for numerical solution of linear and nonlinear differential equations. HWCM is a simple and straightforward method. To construct better approximation we switch from Haar wavelet to Legendre wavelet basis. Superior accuracy is attained in the case of LWCM.

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