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A SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS

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Abstract. Complex-valued harmonic that are univalent and sense preserving in the open unit disc U can be written in the form $f = h + \bar{g}$ where h and g are analytic in U. In this parer authors introduce the class

$$RS(m, n, \alpha), (0 \leq \alpha < 1, m \in N, n \in N_O, m > n)$$

consisting of harmonic univalent functions $f = h + \bar{g}$. for which

$$\sum_{k=2}^{\infty} k(m-n)(|a_n| + |b_n|) \leq (1-\alpha)(1-|b_1|), |b_1| < 1.$$

We obtain distortion bounds extreme points for functions belonging to this class and we study some features of subclasses of this class. We also show that class studied in this paper is closed under convolution and convex combinations.

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1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain D if both u and v are real harmonic in D.
In any simply connected domain we can write $f = h + \bar{g}$ where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f.
A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$. See Clunie and Shell-Small[1]
Denote by S_H the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disc U for which $f(0) = f_z(0) - 1 = 0$.
Then for $f = h + \bar{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1, (1.1)$$

note that S_H reduces to the class of normalized analytic univalent function if the co-analytic part of its member is zero.
A function of the form (1.1), is harmonic starlike for $|z| = r < 1$. If

$$Re\frac{zh'(z) - \overline{zg'(z)}}{h(z) + g(z)} > 0, (1.2)$$

Silverman in [2] proved that the coefficient condition $\sum_{k=2}^{\infty} k(|a_k|+|b_k|) \leq 1$ is sufficient condition for function $f = h + \bar{g}$ to be harmonic starlike.

2. The class $RS(m, n, \alpha)$

Denote by $RS(m, n, \alpha)$ the class of all harmonic function of the form (1.1) that satisfy the condition

$$\sum_{k=2}^{\infty} k(m-n)(|a_k| + |b_k|) \leq (1-\alpha)(1-|b_1|), (2.1)$$

where $m \in N$, $n \in N_0$, $m > n$, $(0 \leq \alpha < 1)$ and $|b_1| < 1$
The class $RS(m, n, \alpha)$ with $b_1 = 0$ will be denoted by $RS^0(m, n, \alpha)$ and suppose that $RS(1, 0, \alpha) \equiv RS(\alpha)$
If h,g and H,G are the form (1.1) and if $f = h+\bar{g}$, $F = H+\bar{G}$ then the convolution of f and F is defined to be the function

$$(f * F) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \overline{\sum_{k=1}^{\infty} b_k B_k z^k}, (2.2)$$

while the integral convolution is defined by

$$(f \Diamond F)(z) = z + \sum_{k=2}^{\infty} \frac{a_k A_k}{k} z^k + \overline{\sum_{K=1}^{\infty} \frac{b_k B_k}{k} z^k}, (2.3)$$

See [3]

The δ-neighborhood of f is the set:

$$N_\delta(f) = \{F, \sum_{k=2}^{\infty} k(|a_k - A_k| + |b_k - B_k|) + |b_1 - B_1| \leq \delta\}, (2.4)$$

.
See $[4-7]$.

3. Main results

First let us give the interrelation between the class $RS(m,n,\alpha_1)$ and $RS(m,n,\alpha_2)$ where $(0 \leq \alpha_1 \leq \alpha_2 < 1)$.

Theorem 1. *$RS(m,n,\alpha_2) \subseteq RS(m,n,\alpha_1)$ where $(0 \leq \alpha_1 \leq \alpha_2 < 1)$. Consequently $RS^0(m,n,\alpha_2) \subseteq RS^0(m,n,\alpha_1)$. In particular $RS(m,n,\alpha) \subseteq RS(m,n,0)$ and $RS^0(m,n,\alpha) \subseteq RS^0(m,n,0)$.*

Proof. Let $f \in RS(m,n,\alpha_2)$ we have

$$\sum_{k=2}^{\infty} k(m-n)(|a_n|+|b_n|) \leq (1-\alpha_2)(1-|b_1|) \leq (1-\alpha_1)(1-|b_1|)$$

Thus $f \in RS(m,n,\alpha_1)$.
The proof of theorem is complete.

Theorem 2. *The class $RS(m,n,\alpha)$ consisting of univalent sense-preserving harmonic mappings.*

Proof. If $z_1 \neq z_2$ then

$$|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}| \geq 1 - |\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}| >$$

$$1 - \frac{|z_1 - z_2| \sum_{k=1}^{\infty} k|b_k|}{|z_1 - z_2|(1 - \sum_{k=2}^{\infty} k|a_k|)} \geq$$

$$1 - \frac{\sum_{k=1}^{\infty} k(m-n)|b_k|}{1 - \sum_{k=2}^{\infty} k(m-n)|a_k|} \geq 0$$

which proves univalence .
Note that f is sense-preserving in U because

$$|h'(z)| \geq 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} > \sum_{k=1}^{\infty} \frac{k(m-n)}{1-\alpha}|b_k||z|^{k-1} \geq \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|$$

Therefore f is sense-preserving.

Theorem 3. $RS(\alpha) \subseteq RH(m, n, \alpha)$, $\forall m \in N, \forall n \in N_0$

Proof. Let $f \in RS(\alpha)$
Since $\sum_{K=2}^{\infty} k(|a_k| + |b_k|) \leq \sum_{k=2}^{\infty} k(m-n)(|a_k| + |b_k|)$.
Thus $f \in RS(m, n, \alpha)$.

Corollary 4. *If $f \in RS(\alpha)$ then f is harmonic starlike.*

Proof. By hypothesis we have

$$\sum_{k=2}^{\infty} k(|a_k| + |b_k|) \leq (1-\alpha)(1-|b_1|) \leq 1$$

Thus f is harmonic starlike.

Theorem 5. *If $f \in RS(m, n, \alpha)$ then*

$$|f(z)| \leq r(1+|b_1|) + \frac{(1-\alpha)(1-|b_1|)r^2}{2(m-n)}$$

and

$$|f(z)| \geq r(1-|b_1|) - \frac{(1-\alpha)(1-|b_1|)r^2}{2(m-n)}$$

The bonds are sharp for the functions

$$f(z) = z + |b_1|\bar{z} + \frac{(1-\alpha)(1-|b_1|)}{2(m-n)} z^2$$

and

$$f(z) = z + |b_1|\bar{z} + \frac{(1-\alpha)(1-|b_1|)}{2(m-n)} \bar{z}^2$$

Proof. Let $f \in RS(m, n.\alpha)$. Taking the absolute value of f
we have

$$|f(z)| = |z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=2}^{\infty} b_k Z^k}| \leq$$

$$|z| + \sum_{k=2}^{\infty} |a_k||z^k| + \sum_{k=2}^{\infty} |b_k||z|^k + |b_1||z| \leq$$

$$r(1+|b_1|) + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^2 \leq$$

$$r(1+|b_1|)+\frac{1}{2(m-n)}\sum_{k=2}^{\infty}k(m-n)(|a_k|+|b_k|)r^2 \leq$$
$$r(1+|b_1|)+\frac{(1-\alpha)(1-|b_1|)}{2(m-n)}r^2$$

and

$$|f(z)| \geq (1+|b_1|)|z|-\sum_{k=2}^{\infty}(|a_k|+|b_k|)|z|^k \geq$$
$$(1-|b_1|)r-\frac{1}{2(m-n)}\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2 \geq$$
$$(1-|b_1|)r-\frac{(1-\alpha)(1-|b_1|)}{2(m-n)}r^2$$

the functions $z+|b_1|\bar{z}+\frac{(1-\alpha)(1-|b_1|)}{2(m-n)}z^2$ and $z+|b_1|\bar{z}+\frac{(1-\alpha)(1-|b_1|)}{2(m-n)}\bar{z}^2$ show that the bounds given in theorem are sharp.

Corollary 6. *If $f \in RS(m,n,\alpha)$ then*

$$\{w : |w| < \frac{(1-|b_1|)(2(m-n)-(1-\alpha))}{2(m-n)} \subseteq f(U)\}$$

Next we determine the extreme points of closed convex hulls of, $RS^0(m,n,\alpha)$,denoted by clco $RS^0(m,n,\alpha)$

Theorem 7. *Set $h_1(z)=z$, $h_k(z)=z+\frac{(1-\alpha)}{k(m-n)}z^k$ $(k=2,3,...)$*
$g_k(z)=z+\frac{(1-\alpha)}{k(m-n)}\bar{z}^k$ $(k=2,3,...)$ then $f \in clcoRS^0(m,n,\alpha)$
if and only if it can be expressed in the form

$$f(z)=\sum_{k=1}^{\infty}\lambda_k h_k+\gamma_k g_k, (3.1)$$

where $\lambda_k \geq 0$, $\gamma_k \geq 0$, $\lambda_1 = 1-\sum_{k=2}^{\infty}(\lambda_k+\gamma_k) \geq 0$, $\gamma_1 = 0$
In particular the extreme point of $RS^0(m,n,\alpha)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For function f of the form (3.1) write
$f(z) = \sum_{k=1}^{\infty} \lambda_k h_k + \gamma_k g_k = z + \sum_{k=2}^{\infty} \lambda_k (\frac{1-\alpha}{k(m-n)}) z^k + \sum_{k=2}^{\infty} \gamma_k (\frac{1-\alpha}{k(m-n)}) \bar{z}^k$
then

$$\sum_{k=2}^{\infty} \frac{k(m-n)}{(1-\alpha)} \frac{(1-\alpha)}{k(m-n)} (\lambda_k + \gamma_k) = \sum_{k=2}^{\infty} (\lambda_k + \gamma_k) = 1 - \lambda_1 \leq 1$$

Thus $f \in clcoRS^0(m, n, \alpha)$.
Conversely : Set

$$\lambda_k = \frac{k(m-n)}{1-\alpha} |a_k|$$
$$(k = 2, 3, ...)$$

,

$$\gamma_k = \frac{k(m-n)}{1-\alpha} |b_k|$$
$$(k = 2, 3, ...)$$

and $\gamma_1 = 0$.
We note that $(0 \leq \lambda_k \leq 1)$, $(0 \leq \gamma_k \leq 1)$ according definition of (λ_1) we have

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} (\lambda_k + \gamma_k)$$

note that by condition (2.1) $\lambda_1 \geq 0$, Consequently we have obtain

$$f(z) = \sum_{k=1}^{\infty} \lambda_k h_k + \gamma_k g_k$$

as required.

Theorem 8. *The class $RS(m, n, \alpha)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, ...$ let $f_i(z) \in RS(m, n, \alpha)$, where $f_i(z)$ is given by

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{k_i} z^k + \sum_{k=2}^{\infty} k(m-n)(|a_{k_i}| + |b_{k_i}|) \leq (1-\alpha)(1 - |b_1|).$$

For $\sum_{i=1}^{\infty} t_i = 1$ $(0 \leq t_i \leq 1)$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{k=2}^{\infty}(\sum_{i=1}^{\infty} t_i a_{k_i}) z^k + \overline{\sum_{k=2}^{\infty}(\sum_{i=1}^{\infty} t_i b_{k_i}) z^n}$$

then we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} k(m-n)(\sum_{i=1}^{\infty} t_i |a_{k_i}| + \sum_{i=1}^{\infty} t_i |b_{k_i}|) \\
&= \sum_{i=1}^{\infty} t_i (\sum_{k=2}^{\infty} k(m-n)|a_{k_i}| + |b_{k_i}|) \leq (1-\alpha)(1-|b_1|) \sum_{i=1}^{\infty} t_i \\
&= (1-\alpha)(1-|b_1|)
\end{aligned}$$

Therefore $\sum_{i=1}^{\infty} t_i f_i \in RS(m, n, \alpha)$.

Theorem 9. *For* $(0 \leq \alpha_1 \leq \alpha_2 < 1)$ *let* $f \in RS(m, n, \alpha_1)$ *and*
$F \in RS(m, n.\alpha_2)$
Then $f * F \in RS(m, n, \alpha_2) \subseteq RS(m, n, \alpha_1)$

Proof. Let

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \in RS(m, n, \alpha_1)$$

and

$$F(z) = z + \sum_{k=2}^{\infty} A_k z^k + \overline{\sum_{k=1}^{\infty} B_k z^K} \in RS(m, n, \alpha_2)$$

then

$$(f * F)(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \overline{\sum_{k=1}^{\infty} b_k B_k z^k}$$

We wish to show that coefficients of $f * F$ satisfy the required condition given in (2.1).
For $f \in RS(m, n, \alpha_1)$ we note that $|a_k| \leq 1$ and $|b_k| \leq 1$.
Now for the convolution function $f * F$ we have

$$\sum_{k=2}^{\infty} k(m-n)(|a_k A_k| + |b_k B_k|) \leq \sum_{k=2}^{\infty} k(m-n)(|A_k| + |b_k|) \leq (1-\alpha_2)(1-|b_1|)$$

Thus $f * F \in RS(m,n,\alpha_2) \subseteq RS(m,n,\alpha_1)$.
Let K_H^0 denote the class of harmonic univalent function of the form (1.1) with $b_1 = 0$ that map U onto convex domain.It is known $[2, Theorem\ 5.10]$ that the sharp inequalities $|A_K| \leq \frac{(k+1)}{2}$, $|B_k| \leq \frac{(k-1)}{2}$ are true.

These result will be used in the next theorem.

Theorem 10. *Suppose that* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=2}^{\infty} b_k z^k}$ *and* $f(z) \in RS^0(m,n,\alpha)$ *and also* $F(z) = z + \sum_{k=2}^{\infty} A_k z^k + \overline{\sum_{k=2}^{\infty} B_k z^k}$ *belong to* K_H^0 *then*

$$f \Diamond F(z) = z + \sum_{k=2}^{\infty} \frac{a_k A_k}{k} z^k + \overline{\sum_{K=1}^{\infty} \frac{b_k B_k}{k} z^k} \in RS^0(m.n,\alpha)$$

Proof. Since $f \in RS^0(m,n,\alpha)$ then $\sum_{k=2}^{\infty} k(m-n)(|a_k| + |b_k|) \leq (1-\alpha)$ we have

$$\sum_{k=2}^{\infty} k(m-n)|\frac{a_k A_k}{k}| + |\frac{b_k B_k}{k}| \leq$$

$$\sum_{k=2}^{\infty} k(m-n)(|a_k||\frac{k+1}{2k}| + |b_k||\frac{k-1}{2k}|) \leq$$

$$\sum_{k=2}^{\infty} (|a_k| + |b_k|) \leq (1-\alpha)$$

Therefore $f \Diamond F \in RS^0(m,n,\alpha)$.
Let P_H^0 denote the class of functions F complex and harmonic in U.
$F = H + \bar{G}$ such that $\mathrm{Re}F(z) > 0$ and

$$H(z) = 1 + \sum_{k=1}^{\infty} A_k z^k, G(z) = \sum_{k=2}^{\infty} B_k z^k$$

it is known $[8, Theorem\ 3\ ]$ that the sharp inequalities
$|A_k| \leq k+1$, $|B_k| \leq k-1$.

Theorem 11. *Suppose that* $f \in RS^0(m,n,\alpha)$ *and*

$$F(z) = 1 + \sum_{k=1}^{\infty} A_k z^k + \overline{\sum_{k=2}^{\infty} B_k z^k}$$

belong to P_H^0*, if* $T \geq 2$ *then* $\frac{1}{T}(f \lozenge F)$ *in* $RS^0(m,n,\alpha)$

Proof. Since $f \in RS^0(m,n,\alpha)$ then we have

$$\sum_{k=2}^{\infty} k(m-n)(|a_k| + |b_k|) \leq (1-\alpha)$$

now using the (2.3) we have

$$\sum_{k=2}^{\infty} k(m-n)|\frac{a_k A_k}{Tk}| + |\frac{b_k B_k}{Tk}| \leq$$

$$\sum_{k=2}^{\infty} k(m-n)(|a_k| + |b_k|) \leq (1-\alpha)$$

Thus $\frac{1}{T}(f \lozenge F) \in RS^0(m,n,\alpha)$.

Theorem 12. *Let* $f \in RS^0(m,n,\alpha)$ *and* $\delta \leq \alpha$ *if* $F \in N_\delta(f)$ *then* F *is harmonic starlike*

Proof. Let $F(z) = z + \sum_{k=2}^{\infty} A_k z^n + \overline{\sum_{k=2}^{\infty} B_n z^n}$ belong to $RS^0(m,n,\alpha)$
we have

$$\sum_{k=2}^{\infty} k(|A_k| + |B_k|) \leq \sum_{k=2}^{\infty} k(|A_k - a_k| + |B_k - b_k|) + \sum_{k=2}^{\infty} k(|a_k| + |b_k|) \leq$$

$\delta + (1-\alpha) \leq 1$
Thus $F(z)$ is harmonic starlike function.

References

[1] J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Series A. I Math. 9 (1984) 3-25

[2] H. Silverman, *Harmonic univalent functions with negative coefficients*, J. Math. Anal. Appl. 220 (1998) 283-289

[3] K.K. Dixit and S. Porwal, *On a subclass of harmonic unnivalent functions*, Journal of inequalities in pure and applied mathematics.Volume 10 (2009).Issue 1,Article 27, 9pp

[4] M. Özturk and S. Yalcin. *On univalent harmonic functions*, J. Inequal. in Pure and Appl. Math, 3(4) (2002), art.61.

[5] O. Altintas, Ö. Özkan and H.M. Srivastava,*Neighborhoods of a class of analytic functions with negative coefficients*, Appl. Math, Lett,13 (2000) 63-67.

[6] St. Rusheweyh.*Neighborhoods of univalent functions.* Proc. Amer. Math. Soc. 81. (1981) 521-528.

[7] Y. Avci and E. Zlotkiewicz.*On harmonic univalent mappings*, Ann. Universitatis Maria Curie-Sklodowska. XLIV (1) (1990), 1-7.

[8] Z.J. Jakubowski, W. Majchrzak and K. Skalska. *Harmonic mappings with a positive real part*, Materialy XIV Konferenji z Teorii Zagadnien Ekstremalnych lodz. (1993) 17-24.

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