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ADOMIAN-TAU OPERATIONAL METHOD FOR SOLVING NON-LINEAR FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS WITH PADE APPROXIMANT

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Abstract. In this paper we develop a new method to find numerical solution for the Non Linear Fredholm Integro-Differential Equations with pade approximant (NFIDEP). To this end, we present our method based on the matrix form of NFIDEP. The corresponding unknown coefficients of the approximate solution will be determined by using computational aspects of some special matrices. Finally we illustrate accuracy and convergence of the presented method by presenting some numerical examples.

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Keywords: Fredholm Integro-Differential Equations, Matrix Forms, Numerical Solutions, Pade approximate solution.

1. Introduction

Recently Adomian decomposition method and operational approach of the tau method have been developed for solving various types of differential, integral and integro differential equations [8,10]. The Tau method developed in [2-4] for the numerical solution of linear Fredholm and Volterra integral and integro-differential equations. The object of this paper is to present a similar operational approach for the numerical solution of non-linear Fredholm integro-differential equations of the second kind with initial conditions. This method leads to an algorithm with remarkable simplicity, while retaining the accuracy of results.

2. Non-linear Fredholm integro-differential equations

Consider a non-linear Fredholm integro-differential equation of the form

$$Dy(x) - \lambda \int_0^a k(x,t)F(y(t))dt = f(x), \quad x \in [0,a] \tag{1}$$

with the given initial conditions

$$y^{(j)}(0) = d_j, \quad j = 0, 1, \cdots, n_d - 1 \tag{2}$$

where D is a linear differential operator of order n_d with polynomial coefficients $p_i(x)$, that is

$$D = \sum_{i=0}^{n_d} p_i(x) \frac{d^i}{dx^i}, \quad p_i(x) = \sum_{j=0}^{\alpha_i} p_{ij} x^j. \tag{3}$$

Assume that $f(x)$ and $k(x,t)$ in (1) are polynomials, otherwise they can be approximated by polynomials to any degree of accuracy (by Lagrange interpolation, Taylor series or any other suitable method). Moreover, suppose that $y_n(x)$ be the Tau method approximation of degree n for $y(x)$. Then we can write

$$p_i(x) = \sum_{j=0}^{\alpha_i} p_{ij} x^j = \underline{p}_i \underline{\underline{X}}$$

$$f(x) = \sum_{j=0}^{n} f_j x^j = \underline{f} \underline{\underline{X}}$$

$$k(x,t) = \sum_{i=0}^{n} \sum_{j=0}^{n} k_{ij} x^i t^j$$

$$y_n(x) = \sum_{j=0}^{n} a_j x^j = \underline{a}_n \underline{\underline{X}} \tag{4}$$

where $\underline{p}_i = [p_{i0}, \cdots, p_{i,\alpha_i}, 0, \cdots]$, $\underline{f} = [f_0, \cdots, f_n, 0, \cdots]$, $\underline{a}_n = [a_0, \cdots, a_n, 0, \cdots]$ and $\underline{\underline{X}} = [1, x, x^2, \cdots]^T$ are respectively the coefficients vectors of $p_i(x)$, right-hand side of equation (1), unknown coefficients vector and the basis vector. Without loss of generality we have taken all polynomials of degree n, because if $f(x)$, $k(x,t)$, and $y_n(x)$ are respectively of different degrees n_f, (n_x,n_t) and n_y then we can set $n = \max\{n_f, n_x, n_t, n_y\}$.

3. Matrix representation for $Dy(x)$

The effect of differentiation or shifting on the coefficients $\underline{a}_n = [a_0, a_1, \cdots, a_n, 0, \cdots]$ of a polynomial $y_n(x) = \underline{a}_n \underline{\underline{X}}$ is the same as that of post-multiplication of $\underline{a}_n$ by either the matrix η or the matrix μ defined by

$$\mu = \begin{bmatrix} 0 & 1 & 0 & 0 & \\ & 0 & 1 & 0 & \\ & & 0 & 1 & \vdots \\ & & & 0 & \\ & & \dots & & \ddots \end{bmatrix} \quad \text{and} \quad \eta = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 2 & 0 & & \vdots \\ 0 & 0 & 3 & 0 & \\ & & \dots & & \ddots \end{bmatrix}.$$

Lemma 1. *Let $y_n(x)$ be a polynomial of the following form:*

$$y_n(x) = \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{\infty} a_i x^i.$$

Then we have

$$i)\ \ \frac{d^r}{dx^r} y_n(x) = \underline{a}_n \eta^r \underline{\underline{X}}\ , \quad r = 1, 2, 3, \cdots$$

$$ii)\ x^r y_n(x) = \underline{a}_n \mu^r \underline{\underline{X}}\ , \quad r = 1, 2, 3, \cdots$$

where $\underline{a}_n = [a_0, a_1, \dots, a_n, 0, \dots]$.

The proof follows immediately by induction.□

With the above notations we state the following theorem.

Theorem 2. *If the operator D and the polynomial $y_n(x)$ are of the forms (3),(4) then $Dy_n(x) = \underline{a}_n \Pi \underline{\underline{X}}$, where*

$$\Pi = \sum_{i=0}^{n_d} \eta^i p_i(\mu).$$

Proof. See [11].

The structure of the matrix Π is as follows

$$\Pi = \begin{bmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \cdots & \pi_{1,m_1} & 0 & & & & \\ \vdots & & & \vdots & & & \ddots & & & \\ \pi_{n_d+1,1} & \pi_{n_d+1,2} & \pi_{n_d+1,3} & \cdots & & \pi_{n_d+1,m_{n_d+1}} & & 0 & & \\ 0 & \pi_{n_d+2,2} & \pi_{n_d+2,3} & \cdots & & & & \pi_{n_d+2,m_{n_d+2}} & 0 & \\ & \ddots & & & & & & & & \ddots \end{bmatrix}$$

where

$$m_i = \begin{cases} \max\{\alpha_0 + i, \alpha_1 + i - 1, \cdots, \alpha_{i-1} + 1\} & if \quad i = 1, 2, \cdots, n_d + 1 \\ m_{n_d+1} + i - n_d + 1 & if \quad i = n_d + 2, n_d + 3, \cdots \end{cases}$$

and

$$\pi_{i,j} = \sum_{k=0}^{i-1} \frac{(i-1)!}{(i-1-k)!} \hat{p}_{k,j-i+k} \quad i = 1, 2, \cdots \quad j = 1, 2, \cdots, m_i$$

$$\hat{p}_{i,j} = \begin{cases} p_{i,j} & if \quad j = 0, 1, \cdots, \alpha_i \\ 0 & if \quad j < 0 \ \ or \ \ j > \alpha_i. \end{cases}$$

4. Matrix representation for the Fredholm integral term

If we replace $y(x)$ in (1) by $y_n(x) = \sum_{i=0}^{n} a_i x^i$, we will have:

$$Dy_n(x) - \lambda \int_0^a k(x,t) F(y_n(t)) dt = f(x), \quad x \in [0, a]. \tag{5}$$

Now, we expand $F(y_n(x))$ around a_0, since a_0 is determined from (5), whenever it is an integral equation and from (2) (for $j = 0$) whenever it is an integro-differential equation.

$$\begin{aligned} F(y_n(x)) &= \sum_{l=0}^{\infty} \frac{F^{(l)}(a_0)}{l!} (\sum_{j=1}^{n} a_j x^j)^l \\ &= F(a_0) + \sum_{l=1}^{\infty} (\sum_{k=1}^{l} (\sum_{\substack{j_1 + j_2 + j_3 + \cdots = k \\ j_1 + 2j_2 + 3j_3 + \cdots = l}} \frac{a_1^{j_1} a_2^{j_2} a_3^{j_3} \cdots}{j_1! j_2! j_3! \cdots}) F^{(k)}(a_0)) x^l \\ &= \{F(a_0)\} + \{a_1 F^{'}(a_0)\} x + \{a_2 F^{'}(a_0) + \frac{a_1^2}{2!} F^{''}(a_0)\} x^2 \\ &+ \{a_3 F^{'}(a_0) + a_1 a_2 F^{''}(a_0) + \frac{a_1^3}{3!} F^{'''}(a_0)\} x^3 \\ &+ \{a_4 F^{'}(a_0) + (a_1 a_3 + \frac{a_2^2}{2!}) F^{''}(a_0) + \frac{a_1^2 a_2}{2!} F^{'''}(a_0) + \frac{a_1^4}{4!} F^{(4)}(a_0)\} x^4 + \cdots \\ &= \underline{\mathcal{F}} \underline{\underline{X}} \end{aligned}$$

where
$\underline{\mathcal{F}} = [\mathcal{F}_0, \mathcal{F}_1, \cdots]$ and $\mathcal{F}_j = \mathcal{F}(a_0, a_1, \cdots, a_j)$. Therefore

$$\int_0^a k(x,t)F(y_n(t))dt = \underline{\mathcal{F}}_n \underline{K} \underline{\underline{X}} \tag{6}$$

where

$$(\underline{K})_{i,j} = \sum_{q=0}^{\infty} k_{iq} \frac{a^{j+q+1}}{j+q+1} \quad i,j = 0,1,2,\cdots.$$

So that, the unknown coefficients are determined as follows.
From (2) we obtain:

$$a_j = \frac{d_j}{j!} \quad j = 0,1,\cdots,n_d - 1.$$

Other coefficients are determined by using theorem (2), equation (1) and solving the following system of non-linear equations

$$\underline{a}_n \Pi - \lambda \underline{\mathcal{F}}_n \underline{K} = \underline{f}. \tag{7}$$

5. Padé approximants

A Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $u(x)$. The $[L/M]$ Padé approximant to a function $y(x)$ are given by [1].

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)}, \tag{8}$$

Where $P_L(x)$ is polynomial of degree at most L, and $Q_M(x)$ is a polynomial of degree at most M. The formal power series

$$y(x) = \sum_{i=1}^{\infty} a_i x^i,$$

$$y(x) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1}),$$

determine the coefficients of $P_L(x)$ and $Q_M(x)$ by the equating.
Since we can clearly multiply the numerator and denominator by a constant and leave $[L/M]$ unchanged, we imposed the normalization condition

$$Q_M(0) = 1. \tag{9}$$

Finally, we require that $P_L(x)$ and $Q_M(x)$ have noncommon factors. If we write the coefficient of $P_L(x)$ and $Q_M(x)$ as:

$$\begin{aligned} P_L(x) &= p_0 + p_1 x + \cdots + p_L x^L, \\ Q_M(x) &= q_0 + q_1 x + \cdots + q_M x^M, \end{aligned} \tag{10}$$

Then by Eqs. (9) and (10), we may multiply Eq. (8) by $Q_M(x)$, which linearizes the coefficient equations. We can write out Eq. (5) in more details as:

$$\begin{cases} a_{L+1} & +a_L q_1 + \cdots + a_{L-M+1} q_M = 0, \\ a_{L+2} & +a_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0, \\ \vdots & \\ a_{L+M} & +a_{L+M-1} q_1 + \cdots + a_L q_M = 0, \end{cases} \tag{11}$$

$$\begin{cases} a_0 & = p_0, \\ a_1 & +a_0 q_1 = p_1, \\ \vdots & \\ a_L & +a_{L-1} q_1 + \cdots + a_0 q_L = p_L, \end{cases} \tag{12}$$

To solve these equations, we start with Eq.(11), which is a set of linear equations for all the unknown q's. Once the q's are known, then Eq.(12) gives an explicit formula for the unknown p's, which complete the solution. If Eqs.(11) and (12) are nonsingular, then we can solve them directly and obtain Eq.(12), where Eq.(12) holds, and if the lower index on a sum exceeds the upper, the sum is replaced by zero:

$$\left[\frac{L}{M}\right] = \frac{\begin{vmatrix} a_{L-M+1} & \cdots & a_{L+1} \\ \vdots & \ddots & \vdots \\ a_L & \cdots & a_{L+M} \\ \sum\limits_{j=M}^{L} a_{j-M} x^j & \cdots & \sum\limits_{j=0}^{L} a_j x^j \end{vmatrix}}{\begin{vmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{vmatrix}}.$$

Theorem 3. *The* $[L/M]$ *Padé approximant to any formal power series* $y(x)$ *is unique.*

Proof. See [1].

In the Tau-Padé method we use the method of Padé approximation after-treatment method to the solution obtained by the Tau method. This after-treatment method improves the proposed method.

6. Estimation of error function

In this section, an error function is obtained for the approximate solution of Eqs.(1) and (2). Let $e_n(x) = y(x) - y_n(x)$ be called the error function of Tau approximation $y_n(x)$ to $y(x)$ where $y(x)$ is the exact solution. Hence $y_n(x)$ satisfies the following problem:

$$Dy_n(x) - \lambda \int_0^a k(x,t)F(y_n(t))dt = f(x) + H_n(x), \quad x \in [0,a] \tag{13}$$

with

$$y_n^{(j)}(0) = d_j, \quad j = 0, 1, \cdots, n_d - 1. \tag{14}$$

The function $H_n(x)$ is the perturbation term associated with $y_n(x)$. Hence

$$H_n(x) = Dy_n(x) - \lambda \int_0^a k(x,t)F(y_n(t))dt - f(x).$$

We proceed to find an approximation $e_{n,N}(x)$ to the error function $e_n(x)$ in the same way as we did before for the solution of problems in Eqs.(1) and (2). Subtracting Eqs. (13) and (14) from Eqs. (1) and (2) respectively, and taking a term of expansion $F(y(x))$ around $y_n(x)$, the error function $e_n(x)$ satisfies the problem:

$$De_n(x) - \int_0^a k(x,t)\Big(e_n(t)F^{'}(y_n(t)) + \frac{1}{2}e_n^2(t)F^{''}(y_n(t))\Big)dt = -H_n(x), \quad x \in [0,a]$$

with:

$$y_n^{(j)}(0) = 0, \quad j = 0, 1, \cdots, n_d - 1.$$

It should be noted that in order to construct the approximation $e_{n,N}(x)$ to $e_n(x)$, only the right hand side of system (7) needs to be recomputed.

7. Error bound and convergence

In this section, we obtain an error bound for the approximate solution, which implies convergence of the presented method.
Let us define the error function as:

$$e_n(x) = y(x) - y_n(x)$$

where $y(x)$ and $y_n(x)$ are the exact and approximate solutions of Eq. (1), respectively. Then for the analytic function, $y(x)$, it is evident that:

$$|e_n(x)| = |y(x) - y_n(x)| \leq M\frac{|x|^{n+1}}{(n+1)!},$$

where $M = \max\{y^{(n+1)}(x), x \in [0, a]\}$, and so $e_n(x) \to 0$ as $n \to \infty$.

8. Numerical examples

The following examples are given to clarify accuracy of the presented method. Note that all of the presented results obtained by programming in maple 8.

Example 1.

$$\begin{aligned} & xy'(x) - y(x) + \int_0^1 xe^{-2t}y^2(t)dt = (x-1)e^x + x, \quad 0 \leq x \leq 1, \\ & y(0) = 1. \end{aligned}$$

The exact solution is $y(x) = e^x$. For the numerical results with $n = 5, 7$ see Table 1,2.

Example 2.

$$\begin{aligned} & y''(x) - y(x) + \int_0^1 sin(x)e^{3t}y^3(t)dt = sin(x), \quad 0 \leq x \leq 1, \\ & y(0) = 1, \qquad y'(0) = 1. \end{aligned}$$

The exact solution is $y(x) = e^{-x}$. For the numerical results with $n = 5, 7$ see Table 3,4.

In figures 1-2 the absolute errors are compared for the Adomian- Tau and pade approximates.

Remark 1. *Note that in the tables 1-4 and figures 1-2, the notations Exacty, App.y and A.T.Err., Est. Err., pade and Pade Err., denote respectively the exact and approximate solution of the Adomian Tau method, absolute error and absolute estimation error of the Adomian- Tau method, pade approximate solution and absolute error of the pade approximation.*

Remark 2. *Note that the reported results in tables 1-4 show that, by increasing the values of n, the approximate solution is improved and converges to the exact solution which confirm the subject of section 7. The figures 1-2 show that the pade approximation has superiority with respect to the Tau approximation in most cases.*

9. Conclusion

In this paper, we solved a problem in general form which is important in practice and involves general forms of linear and non-linear initial and boundary value problems, general forms of linear and non-linear Volterra integral and integro-differential equations and finally general forms of linear and non-linear Fredholm integral and integro-differential equations. We also designed a remarkably simple algorithm by combining the operation approach of the Tau method and the ADM, which has high accuracy for solving the above mentioned problems and we clarified the accuracy by solving numerical examples (see tables 1-4). Note that the factors that affect the error of this method, may be considered as follows:

1) The number of terms that we use to the approximate solution, which depend on the smoothness of the functions F, K and f.

2) The length of the interval [a,b].

3) The number of digits that we use in computing the numerical results.

Table 1: Example 1

n	x	$Exacty$	$App.y$	$A.T.Err.$	$Est.Err.$	$pade[3,2]$	$PadeErr.$
	0.00	1.000000	0.999964	$3.578163e-05$	$3.578163e-05$	0.999995	$4.810606e-06$
	0.20	1.221403	1.221399	$3.559890e-06$	$1.669428e-06$	1.221401	$2.090922e-06$
	0.40	1.491825	1.491825	$1.186132e-07$	$2.290024e-09$	1.491825	$1.206588e-07$
5	0.60	1.822119	1.822122	$3.607015e-06$	$2.290024e-09$	1.822122	$3.609088e-06$
	0.80	2.225541	2.225548	$6.647873e-06$	$1.669428e-06$	2.225549	$8.175592e-06$
	1.00	2.718282	2.718258	$2.405492e-05$	$3.578163e-05$	2.718291	$8.947726e-06$

Table 2: Example 1

n	x	$Exacty$	$App.y$	$A.T.Err.$	$Est.Err.$	$pade[4,3]$	$PadeErr.$
	0.00	1.000000	1.000000	$1.597301e-07$	$1.597301e-07$	1.000000	$5.477186e-09$
	0.20	1.221403	1.221403	$9.393339e-09$	$2.682853e-09$	1.221403	$6.738161e-09$
	0.40	1.491825	1.491825	$7.614197e-10$	$4.089092e-13$	1.491825	$7.618349e-10$
7	0.60	1.822119	1.822119	$1.341060e-08$	$4.089092e-13$	1.822119	$1.341103e-08$
	0.80	2.225541	2.225541	$2.796576e-08$	$2.682853e-09$	2.225541	$3.084066e-08$
	1.00	2.718282	2.718282	$1.163730e-07$	$1.597301e-07$	2.718282	$5.981338e-08$

Table 3: Example 2

n	x	$Exacty$	$App.y$	$A.T.Err.$	$Est.Err.$	$pade[3,2]$	$PadeErr.$
	0.00	1.000000	0.999987	$1.280141e-05$	$1.280141e-05$	0.999999	$6.513576e-07$
	0.20	0.818731	0.818728	$2.977666e-06$	$5.972626e-07$	0.818728	$2.415279e-06$
	0.40	0.670320	0.670314	$6.036808e-06$	$8.192903e-10$	0.670314	$6.036045e-06$
5	0.60	0.548812	0.548802	$9.736524e-06$	$8.192903e-10$	0.548802	$9.735771e-06$
	0.80	0.449329	0.449315	$1.402418e-05$	$5.972626e-07$	0.449315	$1.348352e-05$
	1.00	0.367879	0.367850	$2.941320e-05$	$1.280141e-05$	0.367861	$1.801470e-05$

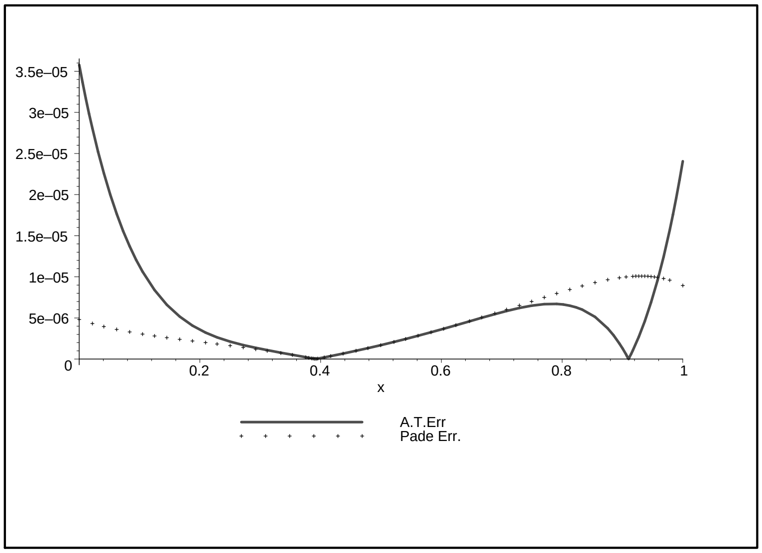

Figure 1: A. T. Err. and pade Err. of example(1) with n=5.

Table 4: Example 2

n	x	$Exacty$	$App.y$	$A.T.Err.$	$Est.Err.$	$pade[5,2]$	$PadeErr.$
	0.00	1.000000	1.000000	$5.755095e-08$	$5.755095e-08$	1.000000	$8.052138e-10$
	0.20	0.818731	0.818731	$1.268779e-08$	$9.666350e-10$	0.818731	$1.172365e-08$
	0.40	0.670320	0.670320	$2.797994e-08$	$1.473304e-13$	0.670320	$2.797979e-08$
7	0.60	0.548812	0.548812	$4.429280e-08$	$1.473304e-13$	0.548812	$4.429266e-08$
	0.80	0.449329	0.449329	$6.156055e-08$	$9.666350e-10$	0.449329	$6.064465e-08$
	1.00	0.367879	0.367879	$1.325714e-07$	$5.755095e-08$	0.367879	$7.898585e-08$

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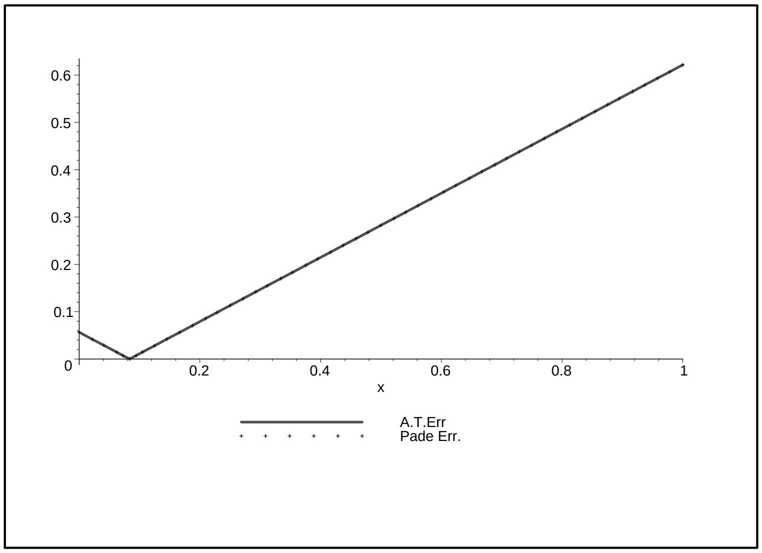

Figure 2: A. T. Err. and pade Err. of example(2) with n=5.

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