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NEW RESULTS RELATED TO STARLIKENESS AND CONVEXITY OF THE BERNARDI INTEGRAL OPERATOR

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Abstract. In this paper we extend some known results related to starlikeness and convexity of the Bernardi integral operator given by

$$L_\beta[f](z) = \frac{\beta+1}{z^\beta}\int_0^z f(t)t^{\beta-1}dt \tag{1}$$

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1. Introduction and preliminaries

Let $\mathcal{H}(\mathcal{U})$ denote the set of holomorphic functions in the open disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let

$$\mathcal{A}_n = \left\{f \in \mathcal{H}(\mathcal{U}) : f(z) = z + a_{n+1}z^{n+1} + ...\right\}$$

with $\mathcal{A}_1 = \mathcal{A}$. Also, for a positive integer n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(\mathcal{U}, f(z) = a + a_n z^n + ..., z \in \mathcal{U}\}$$

and $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathcal{U}\}$.

Let

$$\mathcal{K}(\alpha) = \left\{f \in \mathcal{A} : \text{Re}\frac{zf''(z)}{f'(z)} + 1 > \alpha, z \in \mathcal{U}\right\}$$

denote the class of normalized convex functions of order α, where $\alpha \in \mathbb{R}$, $\alpha < 1$. For $\alpha = 0$, $\mathcal{K}(0) = \mathcal{K}$ denote de class of normalized convex functions in $\mathcal{U}$.

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha, z \in \mathcal{U} \right\}$$

denote de class of starlike function of order α, with $\alpha \in \mathbb{R}$, $\alpha < 1$. For $\alpha = 0$, $\mathcal{S}^*(0) = \mathcal{S}^*$ denote the class of starlike functions in $\mathcal{U}$.

Theorem 1. *[2][10]([7] Theorem 9.5.5., p. 218) If $L_\gamma : \mathcal{A} \to \mathcal{A}$ is the integral operator defined by $L_\gamma[f] = F$, where F is given by*

$$L_\gamma[f](z) = F(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z f(t) t^{\gamma - 1} dt,$$

and $\mathrm{Re}\gamma \geq 0$, *$z \in \mathcal{U}$, then it is well known that:*
(i) $L_\gamma(\mathcal{S}^*) \subset \mathcal{S}^*$;
(ii) $L_\gamma(\mathcal{K}) \subset \mathcal{K}$.

Theorem 2. *([8], Theorem 1) Let $f \in \mathcal{A}$, $\beta \geq 1$ and let*

$$F(z) = L_\beta(z) = \frac{\beta + 1}{z^\beta} \int_0^z f(t) t^{\beta - 1} dt, \ z \in \mathcal{U},$$

If

$$\mathrm{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > -\frac{1}{2\beta}, \ z \in \mathcal{U},$$

then the function F is convex.

Theorem 3. *([9], Theorem 1) Let $f \in \mathcal{A}$, $z \in \mathcal{U}$, $\beta \geq 1$ and*

$$F(z) = L_\beta[f](z) = \frac{\beta + 1}{z^\beta} \int_0^z f(t) t^{\beta - 1} dt, \ z \in \mathcal{U},$$

then the function F is starlike.

2. Main Results

In [9], Georgia Irina Oros was proved that if $f \in \mathcal{S}^*\left(-\frac{1}{2\beta}\right)$, $\beta \geq 1$, then F given by (1) is starlike. We will extend this result from the next theorem:

Theorem 4. *Let $\beta \geq 1$, $f \in \mathcal{A}_n$, $F(z) = L_\beta[f](z)$ where L_β is given by (1).*

If

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > -\frac{\beta}{2},\ z \in \mathcal{U} \tag{2}$$

then

$$\mathrm{Re}\frac{zF'(z)}{F(z)} > -\beta,\ z \in \mathcal{U}.$$

Proof. Since $f \in \mathcal{A}_n$, we have $F(z) = z + b_{n+1}z^{n+1} + ...$, $F(0) = 0$, $F'(0) = 1$.

From (1) we have

$$z^\beta \cdot F(z) = (\beta+1)\int_0^z f(t)t^{\beta-1}dt,\ z \in \mathcal{U}. \tag{3}$$

By differentiating (3) and by a simple calculation we obtain

$$F(z)\left[\beta + \frac{zF'(z)}{F(z)}\right] = (\beta+1)f(z),\ z \in \mathcal{U}. \tag{4}$$

We let

$$p(z) = \frac{1}{\beta+1}\left[\frac{zF'(z)}{F(z)} + \beta\right] = 1 + c_n z^n + ...,\ p(0) = 1,\ p \in \mathcal{H}[1,n]. \tag{5}$$

Using (5), then (4) becomes

$$F(z) \cdot p(z) = f(z),\ z \in \mathcal{U}. \tag{6}$$

By differentiating (6) and using (5), we obtain

$$(1+\beta)p(z) - \beta + \frac{zp'(z)}{p(z)} = \frac{zf'(z)}{f(z)},\ z \in \mathcal{U}. \tag{7}$$

Using (2) and (7), we have

$$\mathrm{Re}\left[(1+\beta)p(z) - \beta + \frac{zp'(z)}{p(z)}\right] = \mathrm{Re}\frac{zf'(z)}{f(z)} > -\frac{\beta}{2}$$

which is equivalent to

$$\mathrm{Re}\left[(1+\beta)p(z) + \frac{zp'(z)}{p(z)} - \frac{\beta}{2}\right] > 0,\ z \in \mathcal{U}. \tag{8}$$

We let $\psi : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C}$,

$$\psi(p(z), zp'(z), z) = (1+\beta)p(z) + \frac{zp'(z)}{p(z)} - \frac{\beta}{2},\ z \in \mathcal{U}. \tag{9}$$

Then (8) is equivalent to

$$\text{Re}\psi(p(z), zp'(z), z) > 0,\ z \in \mathcal{U}. \tag{10}$$

In order to prove our theorem, we use a well known Lemma due to S.S. Miller and P.T. Mocanu (see [3]-[6]). For that we calculate

$$\text{Re}\psi(i\rho, \sigma, z) = \text{Re}\left[(1+\beta)i\rho + \frac{\sigma}{i\rho} - \frac{\beta}{2}\right] = -\frac{\beta}{2} \leq 0.$$

Now, using the above mentioned Lemma, we get that $\text{Re}p(z) > 0$, $z \in \mathcal{U}$, i.e

$$\text{Re}\frac{1}{1+\beta}\left[\frac{zF'(z)}{F(z)} + \beta\right] > 0,$$

which imply that

$$\text{Re}\frac{zF'(z)}{F(z)} > -\beta,\ z \in \mathcal{U}$$

hence $F \in \mathcal{S}^*(-\beta)$, $\beta \geq 0$.

Remark 1. *This result improves the results in Theorem 1.*

Remark 2. *For $\beta = 1$, Theorem 4 extend the results obtained in [7], Theorem 9.5.2, p. 214, (R. J. Libera Theorem) for the Libera operator.*

In [8], Georgia Irina Oros showed that if $f \in \mathcal{K}\left(-\frac{1}{2\beta}\right)$, $\beta \geq 1$, then $F \in \mathcal{K}$, where F is given by (1). We will extend this result by the following theorem:

Theorem 5. *If $\beta \geq 0$, $f \in \mathcal{A}_n$ and satisfies*

$$\text{Re}\frac{zf''(z)}{f'(z)} + 1 > -\frac{\beta}{2} \tag{11}$$

then $L_\beta[f](z) = F(z) \in \mathcal{K}(-\beta)$, where L_β given by (1).

Proof. By, differentiating (3), and by a simple calculation we obtain that

$$F'(z)\left[\beta+1+\frac{zF''(z)}{F'(z)}\right]=(\beta+1)f'(z),\ z\in\mathcal{U}. \tag{12}$$

We let

$$(1+\beta)p(z)=\beta+1+\frac{zF''(z)}{F'(z)}=\beta+1+c_nz^n+...,\ p(0)=1,\ p\in\mathcal{H}[1,n]. \tag{13}$$

Using (13) in (12), we have

$$F'(z)p(z)=f'(z),\ z\in\mathcal{U}. \tag{14}$$

By differentiating (14) and by a simple calculation we obtain

$$\frac{zF''(z)}{F'(z)}+\beta+1+\frac{zp'(z)}{p(z)}=\frac{zf''(z)}{f'(z)}+1+\beta,\ z\in\mathcal{U}. \tag{15}$$

Using (13) in (15) we obtain

$$p(z)+\frac{zp'(z)}{p(z)}=\frac{zf''(z)}{f'(z)}+1+\beta,\ z\in\mathcal{U}. \tag{16}$$

From (11), we have:

$$\mathrm{Re}\left[p(z)+\frac{zp'(z)}{p(z)}-\frac{\beta}{2}\right]>0. \tag{17}$$

We let $\psi:\mathbb{C}^2\times\mathcal{U}\to\mathbb{C}$,

$$\psi(p(z),zp'(z),z)=p(z)+\frac{zp'(z)}{p(z)}-\frac{\beta}{2},\ z\in\mathcal{U}. \tag{18}$$

Then (17) becomes

$$\mathrm{Re}\psi(p(z),zp'(z),z)>0,\ z\in\mathcal{U}. \tag{19}$$

In order to prove our theorem, we use a well known Lemma due to S.S. Miller and P.T. Mocanu (see [3]-[6]). For that we calculate

$$\mathrm{Re}\psi(i\rho,\sigma,z)=\mathrm{Re}\left[i\rho+\frac{\sigma}{i\rho}-\frac{\beta}{2}\right]=-\frac{\beta}{2}<0.$$

Now, using the above mentioned Lemma, we get that $\mathrm{Re}p(z)>0$, $z\in\mathcal{U}$, i.e

$$\mathrm{Re}\left[\frac{zF''(z)}{F'(z)}+1\right] > -\beta,\ z\in\mathcal{U}$$

hence $F\in\mathcal{K}(-\beta)$.

Remark 3. *The results of this theorem extend the results obtained in Theorem 1.*

Remark 4. *For $\beta=1$, the results extend the results of Th. 9.5.2, Th. 9.5.3.[7], p. 214-215.*

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