

FRACTIONAL CALCULUS OF K -BESSEL'S FUNCTION

K.S. GEHLOT, S.D. PUROHIT

ABSTRACT. This paper deals with the study of newly definded special function known as k -Bessel's function [2]. Certain relations that exists between k -Bessel's function and the Riemann-Liouville fractional integrals and differential operators are investigated. It has also been shown that the fractional integrals and differential operators transforms such functions with power multipliers into the function of the same form.

2000 *Mathematics Subject Classification:* 26A33, 33B15, 33C20, 34B30.

Keywords: k -Gamma function, k -Pochhammer symbols, k -Bessel function, fractional calculus.

1. INTRODUCTION

The computation of fractional integrals (or derivatives) of special functions are important from the point of view of the usefulness of these results in the evaluation of generalized integrals, and the solution of differential and integral equations. Fractional integral formulas involving the Bessel function has been developed and play an important role in several physical problems. In fact, Bessel function is an important in studying solutions of differential equations, and they are associated with a wide range of problems in many areas of mathematical physics, like problems of acoustics, radio physics, hydrodynamics, and atomic and nuclear physics. These considerations have led various workers in the field of special functions for exploring the possible extensions and applications for the Bessel function. A useful generalization of the Bessel function called as k -Bessel function has been introduced and studied in [2]. Here we aim at presenting composition formulas of fractional calculus operators and the k -Bessel function.

In 2007, Díaz and Pariguan [1] introduced the generalized k -Gamma Function $\Gamma_k(x)$ as

$$\Gamma_k(x) = \lim_{n \rightarrow \infty} \frac{n!k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \quad (k > 0, x \in \mathbb{C} \setminus k\mathbb{Z}^-), \quad (1)$$

where $(x)_{n,k}$ is the k -Pochhammer symbol and is given by

$$(x)_{n,k} = x(x+k)(x+2k)\dots(x+(n-1)k), x \in \mathbb{C}, k \in \mathbb{R}, n \in \mathbb{N}^+. \quad (2)$$

The integral form of generalized k -Gamma function is given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt, x \in \mathbb{C}, k \in \mathbb{R}, \Re(x) > 0. \quad (3)$$

It follows easily that

$$\Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma\left(\frac{x}{k}\right) \quad (4)$$

and

$$\Gamma_k(x+k) = x\Gamma_k(x). \quad (5)$$

The Beta function ([3, p.19])

$$\int_0^1 t^{n-1}(1-t)^{m-1} dt = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} \quad (n, m > 0). \quad (6)$$

k -Bessel function defined as [2]:

$$J_{\pm\vartheta}^k(z) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{z}{2}\right)^{2r \pm \frac{\vartheta}{k}}}{\Gamma_k(rk \pm \vartheta + k)(r!)}, \quad (7)$$

where $k \in \mathbb{R}^+, \vartheta \in \mathbb{I}$ and $\vartheta > -k$.

The fractional integrals of a function $f(z)$ of order α ([4, Definition 2.1, P.33]) are defined as:

$$(I_{0+}^\alpha f)(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(t)}{(z-t)^{1-\alpha}} dt \quad (z > 0) \quad (8)$$

and

$$(I_-^\alpha f)(z) = \frac{1}{\Gamma(\alpha)} \int_z^\infty \frac{f(t)}{(t-z)^{1-\alpha}} dt \quad (z > 0). \quad (9)$$

Further, the fractional derivatives ([4, Definition 2.2, P. 35]) are defined as follow:

$$\begin{aligned} (D_{0+}^\alpha f)(z) &= \left(\frac{d}{dz}\right)^{[\Re(\alpha)]+1} (I_{0+}^{1-\alpha+[\Re(\alpha)]} f)(z) \\ &= \frac{1}{\Gamma(1-\alpha+[\Re(\alpha)])} \left(\frac{d}{dz}\right)^{[\Re(\alpha)]+1} \int_0^z \frac{f(t)}{(z-t)^{\alpha-[\Re(\alpha)]}} dt \quad (z > 0) \end{aligned} \quad (10)$$

and

$$\begin{aligned} (D_-^\alpha f)(z) &= \left(-\frac{d}{dz}\right)^{[\Re(\alpha)]+1} (I_-^{1-\alpha+[\Re(\alpha)]} f)(z) \\ &= \frac{1}{\Gamma(1-\alpha+[\Re(\alpha)])} \left(-\frac{d}{dz}\right)^{[\Re(\alpha)]+1} \int_z^\infty \frac{f(t)}{(t-z)^{\alpha-[\Re(\alpha)]}} dt \quad (z > 0), \end{aligned} \quad (11)$$

where $\alpha \in \mathbb{C}(\Re(\alpha) > 0)$.

2. FRACTIONAL INTEGRATION OF k -BESSEL'S FUNCTION

In this section, we establish image formulas for the k -Bessel function involving left and right sided fractional integral operators (8) and (9), in term of the same function. These formulas are given by the following theorems:

Theorem 1. *Let $k \in \mathbb{R}^+, a \in \mathbb{R}, \vartheta \in \mathbb{I}, \vartheta > -k$ and $\Re(\alpha) > 0$, then the fractional integral I_{0+}^α of k -Bessel's function (7) is given by*

$$(I_{0+}^\alpha [t^{\frac{\vartheta}{2k}} J_\vartheta^k(a\sqrt{t})])(z) = k^\alpha \left(\frac{a}{2}\right)^{-\alpha} z^{\frac{\vartheta}{2k} + \frac{\alpha}{2}} J_{\vartheta+\alpha k}^k(a\sqrt{z}) \quad (z > 0). \quad (12)$$

Proof. Consider the left hand side (say A) and using the definitions (7) and (8), we have

$$\begin{aligned} A &\equiv (I_{0+}^\alpha [t^{\frac{\vartheta}{2k}} J_\vartheta^k(a\sqrt{t})])(z) \\ &\equiv \frac{1}{\Gamma(\alpha)} \int_0^z \frac{t^{\frac{\vartheta}{2k}}}{(z-t)^{1-\alpha}} \sum_{r=0}^\infty \frac{(-1)^r \left(\frac{a\sqrt{t}}{2}\right)^{2r+\frac{\vartheta}{k}}}{\Gamma_k(rk + \vartheta + k)(r!)} dt. \end{aligned}$$

Interchanging the order of integration and summation, and then evaluating the inner integral by substituting $t = zu$, we obtain

$$A \equiv \frac{1}{\Gamma(\alpha)} \sum_{r=0}^\infty \frac{(-1)^r \left(\frac{a}{2}\right)^{2r+\frac{\vartheta}{k}}}{\Gamma_k(rk + \vartheta + k)(r!)} z^{r+\alpha+\frac{\vartheta}{k}} \int_0^1 u^{r+\frac{\vartheta}{k}} (1-u)^{\alpha-1} du.$$

Now, on using (6) and (4), and rearranging the terms in above expression, we get the following

$$A \equiv k^\alpha \left(\frac{a}{2}\right)^{-\alpha} z^{\frac{\vartheta}{2k} + \frac{\alpha}{2}} J_{\vartheta+\alpha k}^k(a\sqrt{z}),$$

which completes the proof of Theorem 1.

Further theorem involving the right hand sided fractional integration (9) of the k -Bessel's function (7) is given by the following result:

Theorem 2. *Let $k \in \mathbb{R}^+, a \in \mathbb{R}, \vartheta \in \mathbb{I}, \vartheta > -k$ and $\Re(\alpha) > 0$, then the following formula holds true:*

$$(I_-^\alpha [t^{-\frac{\vartheta}{2k}-\alpha-1} J_\vartheta^k\left(\frac{a}{\sqrt{t}}\right)])(z) = k^\alpha \left(\frac{a}{2}\right)^{-\alpha} z^{\frac{\alpha}{2} - \frac{\vartheta}{2k} - 1} J_{\vartheta+\alpha k}^k\left(\frac{a}{\sqrt{z}}\right) \quad (z > 0). \quad (13)$$

Proof. By virtue of the definitions (7) and (9), the left hand side (say B) of above result, we obtain

$$B \equiv (I_-^\alpha [t^{-\frac{\vartheta}{2k}-\alpha-1} J_\vartheta^k\left(\frac{a}{\sqrt{t}}\right)])(z)$$

$$\equiv \frac{1}{\Gamma(\alpha)} \int_z^\infty \frac{t^{-\frac{\vartheta}{2k}-\alpha-1}}{(t-z)^{1-\alpha}} \sum_{r=0}^\infty \frac{(-1)^r \left(\frac{a}{2\sqrt{t}}\right)^{2r+\frac{\vartheta}{k}}}{\Gamma_k(rk+\vartheta+k)(r!)} dt,$$

which on interchanging the order of integration and summation, and evaluating the inner integral by setting $t = \frac{z}{u}$, leads to

$$B \equiv \frac{1}{\Gamma(\alpha)} \sum_{r=0}^\infty \frac{(-1)^r \left(\frac{a}{2}\right)^{2r+\frac{\vartheta}{k}}}{\Gamma_k(rk+\vartheta+k)(r!)} z^{-\frac{\vartheta}{k}-r-1} \int_0^1 u^{1+r+\frac{\vartheta}{k}} (1-u)^{\alpha-1} du.$$

Upon using (6) and (4), we can easily arrive at the right hand side of result (13).

3. FRACTIONAL DIFFERENTIATION OF k -BESSEL'S FUNCTION

We now give images of the k -Bessel's function (7) under the fractional differential operators. Again, we begin with the left hand sided fractional differentiation (10).

Theorem 3. *Let $k \in \mathbb{R}^+$, $a \in \mathbb{R}$, $\vartheta \in \mathbb{I}$, $\vartheta > -k$ and $\Re(\alpha) > 0$, then the fractional differentiation D_{0+}^α of k -Bessel's function is given as under:*

$$(D_{0+}^\alpha [t^{\frac{\vartheta}{2k}} J_\vartheta^k(a\sqrt{t})])(z) = k^{-\alpha} \left(\frac{a}{2}\right)^\alpha z^{\frac{\vartheta}{2k}-\frac{\alpha}{2}} J_{\vartheta-\alpha k}^k(a\sqrt{z}) \quad (z > 0). \quad (14)$$

Proof. Letting $n = [\Re(\alpha)] + 1$, where $[\Re(\alpha)]$ is an integer part of $\Re(\alpha)$, and using the equations (7) and (10), the left hand side of result (14) (say C) reduces to

$$\begin{aligned} C &\equiv (D_{0+}^\alpha [t^{\frac{\vartheta}{2k}} J_\vartheta^k(a\sqrt{t})])(z) \\ &\equiv \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dz}\right)^n \int_0^z \frac{t^{\frac{\vartheta}{2k}}}{(z-t)^{1+\alpha-n}} \sum_{r=0}^\infty \frac{(-1)^r \left(\frac{a\sqrt{t}}{2}\right)^{2r+\frac{\vartheta}{k}}}{\Gamma_k(rk+\vartheta+k)(r!)} dt. \end{aligned}$$

Again, interchanging the order of integration and summation and evaluating the inner integral by using $t = zu$, we obtain

$$C \equiv \frac{1}{\Gamma(n-\alpha)} \sum_{r=0}^\infty \frac{(-1)^r \left(\frac{a}{2}\right)^{2r+\frac{\vartheta}{k}}}{\Gamma_k(rk+\vartheta+k)(r!)} \left(\frac{d}{dz}\right)^n z^{\frac{\vartheta}{k}+r-\alpha+n} \int_0^1 u^{r+\frac{\vartheta}{k}} (1-u)^{n-\alpha-1} du.$$

Now, on using the differential formula, namely

$$\left(\frac{d}{dz}\right)^n z^{\frac{\vartheta}{k}+r-\alpha+n} = \frac{\Gamma\left(\frac{\vartheta}{k}+r-\alpha+n+1\right)}{\Gamma\left(\frac{\vartheta}{k}+r-\alpha+1\right)} z^{\frac{\vartheta}{k}+r-\alpha},$$

and then taking (6) and (4) into account, we get

$$C \equiv k^{-\alpha} \left(\frac{a}{2}\right)^\alpha z^{\frac{\vartheta}{2k}-\frac{\alpha}{2}} J_{\vartheta-\alpha k}^k(a\sqrt{z}).$$

The next result yields the right hand sided fractional differentiation of the k -Bessel's function (7).

Theorem 4. Let $k \in \mathbb{R}^+$, $a \in \mathbb{R}$, $\vartheta \in \mathbb{I}$, $\vartheta > -k$ and $\Re(\alpha) > 0$, then the fractional differentiation D_-^α of k -Bessel's function (7) is given for $z > 0$ by

$$(D_-^\alpha [t^{-\frac{\vartheta}{2k} + \alpha - 1} J_{\vartheta}^k(\frac{a}{\sqrt{t}})])(z) = k^{-\alpha} (\frac{a}{2})^\alpha z^{-\frac{\vartheta}{2k} - \frac{\alpha}{2} - 1} J_{\vartheta - \alpha k}^k(\frac{a}{\sqrt{z}}). \quad (15)$$

Proof. Setting $n = [Re(\alpha)] + 1$, where $[Re(\alpha)]$ is an integer part of $Re(\alpha)$, and using (7) and (11) the left hand side of result (15) (let D), yields to

$$D \equiv \frac{1}{\Gamma(n - \alpha)} (-\frac{d}{dz})^n \int_z^\infty \frac{t^{-\frac{\vartheta}{2k} + \alpha - 1}}{(t - z)^{1 + \alpha - n}} \sum_{r=0}^\infty \frac{(-1)^r (\frac{a}{2\sqrt{t}})^{2r + \frac{\vartheta}{k}}}{\Gamma_k(rk + \vartheta + k)(r!)} dt,$$

which on interchanging the order of integration and summation and evaluating the inner integral by taking $t = \frac{z}{u}$, leads to

$$D \equiv \frac{1}{\Gamma(n - \alpha)} \sum_{r=0}^\infty \frac{(-1)^r (\frac{a}{2})^{2r + \frac{\vartheta}{k}}}{\Gamma_k(rk + \vartheta + k)(r!)} (-\frac{d}{dz})^n z^{n - \frac{\vartheta}{k} - r - 1} \int_0^1 u^{\frac{\vartheta}{k} + r - n} (1 - u)^{n - \alpha - 1} du.$$

Upon using the formula, namely

$$(\frac{d}{dz})^n z^{n - r - 1 - \frac{\vartheta}{k}} = \frac{\Gamma(n - r - \frac{\vartheta}{k})}{\Gamma(-r - \frac{\vartheta}{k})} z^{-r - 1 - \frac{\vartheta}{k}},$$

and definitions (6) and (4), then after certain simplifications we obtain

$$D \equiv \sum_{r=0}^\infty \frac{(-1)^r (\frac{a}{2})^{2r + \frac{\vartheta}{k}} (-1)^n z^{-\frac{\vartheta}{k} - r - 1} \Gamma(n - r - \frac{\vartheta}{k}) \Gamma(\frac{\vartheta}{k} + r - n + 1)}{k^\alpha \Gamma_k(rk + (\vartheta - \alpha k) + k)(r!) \Gamma(\frac{\vartheta}{k} + r + 1) \Gamma(-\frac{\vartheta}{k} - r)}. \quad (16)$$

On the other hand, if we use the reflection formula for the gamma function, namely (see ([3, p. 21]),

$$\Gamma(1 - (-\frac{\vartheta}{k} - r)) \Gamma(-\frac{\vartheta}{k} - r) = \frac{-\pi}{\sin(r\pi + \frac{\vartheta}{k}\pi)} \quad (17)$$

and

$$\Gamma(1 - (n - r - \frac{\vartheta}{k})) \Gamma(n - r - \frac{\vartheta}{k}) = \frac{\pi}{\sin(n\pi - (r + \frac{\vartheta}{k})\pi)}$$

$$\Gamma(1 - (n - r - \frac{\vartheta}{k})) \Gamma(n - r - \frac{\vartheta}{k}) = \frac{\pi}{\sin(n\pi) \cos(r\pi + \frac{\vartheta}{k}\pi) - \cos(n\pi) \sin(r\pi + \frac{\vartheta}{k}\pi)}$$

$$\Gamma\left(1 - \left(n - r - \frac{\vartheta}{k}\right)\right)\Gamma\left(n - r - \frac{\vartheta}{k}\right) = \frac{(-1)^{n+1}\pi}{\sin\left(r\pi + \frac{\vartheta}{k}\pi\right)}, \quad (18)$$

in (16), we obtain

$$D \equiv k^{-\alpha} \left(\frac{a}{2}\right)^{\alpha} z^{-\frac{\vartheta}{2k} - \frac{\alpha}{2} - 1} J_{\vartheta - \alpha k}^k\left(\frac{a}{\sqrt{z}}\right).$$

which completes the proof of Theorem 4.

4. CONCLUSION

The k -generalized Bessel function defined by (7), possess the advantage that the Bessel function, trigonometric functions and hyperbolic functions happen to be the particular cases of this function. Therefore, we conclude this paper with the remark that, the results deduced above are significant and can lead to yield numerous other fractional integrals (derivatives) involving the Bessel function and trigonometric functions by the suitable specializations of arbitrary parameters in the theorems. More importantly, they are expected to find some applications to the solutions of fractional differential and integral equations.

REFERENCES

- [1] R. Díaz, E. Pariguan, *On hypergeometric functions and Pochhammer k -symbol*. Divulgaciones Matemáticas, 15, 2 (2007), 179-192.
- [2] K.S. Gehlot, *Differential equation of k -Bessel's function and its properties*, Nonl. Anal. Diff. Eq., 2, 2 (2014), 61-67.
- [3] Earl D. Rainville, *Special Functions*, The Macmillan Company, New York, 1963.
- [4] S.G. Samok, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives*, Theory and Applications. Gordon and Breach, New York, 1993.

K.S. Gehlot
 Department of Mathematics,
 Government Bangur P.G. College, Pali, Pali-Marwar-306401, India
 email: drksgehl@rediffmail.com

S.D. Purohit
 Department of Basic Sciences (Mathematics),
 College of Technology and Engineering,
 M.P. University of Agri. and Tech., Udaipur-313001, India
 email: sunil_a_purohit@yahoo.com