Acta Universitatis Apulensis
ISSN: 1582-5329

No. 38/2014
pp. 279-296

ORDER OF CONVEXITY OF INTEGRAL TRANSFORMS AND DUALITY

S. Verma, S. Gupta, S. Singh

Abstract. Recently, Ali et al. [2] defined the class $\mathcal{W}_\beta(\alpha,\gamma)$ consisting of functions f which satisfy

$$\Re e^{i\phi}\left((1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma zf''(z)-\beta\right)>0,$$

for all $z\in E=\{z:|z|<1\}$ and for $\alpha,\gamma\geq 0$ and $\beta<1$, $\phi\in\mathbb{R}$ (the set of reals). For $f\in\mathcal{W}_\beta(\alpha,\gamma)$, they discussed the convexity of the integral transform

$$V_\lambda(f)(z):=\int_0^1\lambda(t)\frac{f(tz)}{t}dt,$$

where λ is a non-negative real-valued integrable function satisfying the condition $\int_0^1\lambda(t)dt=1$. The aim of present paper is to find conditions on $\lambda(t)$ such that $V_\lambda(f)$ is convex of order δ $(0\leq\delta\leq 1/2)$ whenever $f\in\mathcal{W}_\beta(\alpha,\gamma)$. As applications, we study various choices of $\lambda(t)$, related to classical integral transforms.

2000 *Mathematics Subject Classification*: 30C45, 30C80.

Keywords: Starlike function, Convex function, Hadamard product, Duality.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f defined in the open unit disc $E=\{z:|z|<1\}$ with the normalization $f(0)=f'(0)-1=0$. Let $\mathcal{A}_0=\{g:\ g(z)=f(z)/z,\ f\in\mathcal{A}\}$. Let S be the subclass of $\mathcal{A}$ consisting of univalent functions in E. A function $f\in S$ is said to be starlike or convex, if f maps E conformally onto the domains, respectively, starlike with respect to the origin and convex. The generalization of these two classes are given by the following analytic characterizations :

$$S^*(\beta)=\left\{f\in\mathcal{A}:\Re\left(\frac{zf'(z)}{f(z)}\right)>\beta,\quad 0\leq\beta<1\right\}$$

$$K(\beta) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta, \quad 0 \le \beta < 1 \right\}.$$

For $\beta = 0$, we usually set $S^*(0) = S^*$ and $K(0) = K$.

For two functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ and $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$ in $\mathcal{A}$, their Hadamard product (or convolution) is the function $f * g$ defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

For $f \in \mathcal{A}$, Fournier and Ruscheweyh [8] introduced the operator

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt, \tag{1}$$

where λ is a non-negative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t)dt = 1$. This operator contains some of the well-known operators such as Libera, Bernardi and Komatu as its special cases. This operator has been studied by a number of authors for various choices of $\lambda(t)$ (for example see [1], [4], [6], [8]). Fournier and Ruscheweyh [8] applied the duality theory ([10, 11]) to prove the starlikeness of the linear integral transform $V_\lambda(f)$ when f varies in the class

$$\mathcal{P}(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} | \Re e^{i\phi} \left(f'(z) - \beta \right) > 0, \ z \in E \right\}.$$

In 1995, Ali and Singh [3] discussed the convexity properties of the integral transform (1) for functions f in the class $\mathcal{P}(\beta)$. In 2002, Choi et al. [7] investigated convexity properties of the integral transform (1) for functions f in the class

$$\mathcal{P}_\gamma(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} | \Re e^{i\phi} \left((1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - \beta \right) > 0, \ z \in E \right\}.$$

It is evident that the class $\mathcal{P}_\gamma(\beta)$ is closely related to the class $\mathcal{R}_\gamma(\beta)$ defined by

$$\mathcal{R}_\gamma(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} | \Re e^{i\phi} \left(f'(z) + \gamma z f''(z) - \beta \right) > 0, \ z \in E \right\}.$$

Clearly, $f \in \mathcal{R}_\gamma(\beta)$ if and only if zf' belongs to $\mathcal{P}_\gamma(\beta)$.

In a very recent paper, R. M. Ali et al. [2] discussed the convexity of the integral transform (1) for the functions f in a more general class $\mathcal{W}_\beta(\alpha, \gamma)$ given by

$$\left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} | \Re e^{i\phi} \left((1 - \alpha + 2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z) - \beta \right) > 0, \ z \in E \right\}. \tag{2}$$

Note that $\mathcal{W}_\beta(1,0) \equiv \mathcal{P}(\beta)$, $\mathcal{W}_\beta(\alpha,0) \equiv \mathcal{P}_\alpha(\beta)$ and $\mathcal{W}_\beta(1+2\gamma,\gamma) \equiv \mathcal{R}_\gamma(\beta)$.

In the present paper, we shall mainly tackle the problem of finding a sharp estimate of the parameter β that ensures $V_\lambda(f)$ to be convex of order δ for $f \in \mathcal{W}_\beta(\alpha,\gamma)$. To prove our result, we shall need the duality theory for convolutions, so we include here some basic concepts and results from this theory. For a subset $\mathcal{B} \subset \mathcal{A}_0$, we define

$$\mathcal{B}^* = \{g \in \mathcal{A}_0 : \ (f * g)(z) \neq 0, \ z \in E, \text{ for all } f \in \mathcal{B}.\}$$

The set $\mathcal{B}^*$ is called the dual of $\mathcal{B}$. Further, the second dual of $\mathcal{B}$ is defined as $\mathcal{B}^{**} = (\mathcal{B}^*)^*$. We state below a fundamental result.

Theorem 1. *Let*

$$\mathcal{B} = \left\{ \beta + (1-\beta)\left(\frac{1+xz}{1+yz}\right) : \ |x| = |y| = 1 \right\}, \ \beta \in \mathbb{R}, \ \beta \neq 1.$$

Then, we have

1. $\mathcal{B}^{**} = \big\{ g \in \mathcal{A}_0 : \ \exists \phi \in \mathbb{R} \text{ such that} \Re\{e^{i\phi}(g(z)-\beta)\} > 0, \ z \in E \big\}$.
2. *If* Γ_1 *and* Γ_2 *are two continuous linear functionals on* $\mathcal{B}$ *with* $0 \notin \Gamma_2$*, then for every* $g \in \mathcal{B}^{**}$ *we can find* $v \in \mathcal{B}$ *such that*

$$\frac{\Gamma_1(g)}{\Gamma_2(g)} = \frac{\Gamma_1(v)}{\Gamma_2(v)}.$$

The basic reference to this theory is the book by Ruscheweyh [10] (see also [11]).

2. Preliminaries

We follow the notations used in [1]. Let $\mu \geq 0$ and $\nu \geq 0$ satisfy

$$\mu + \nu = \alpha - \gamma \quad \text{and} \quad \mu\nu = \gamma. \tag{3}$$

When $\gamma = 0$, then μ is chosen to be 0, in which case, $\nu = \alpha \geq 0$. When $\alpha = 1 + 2\gamma$, (3) yields $\mu + \nu = 1 + \gamma = 1 + \mu\nu$, or $(\mu - 1)(1 - \nu) = 0$.

(i) For $\gamma > 0$, then choosing $\mu = 1$ gives $\nu = \gamma$.
(ii) For $\gamma = 0$, then $\mu = 0$ and $\nu = \alpha = 1$.

Whenever the particular case $\alpha = 1+2\gamma$ will be considered, the values of μ and ν for $\gamma > 0$ will be taken as $\mu = 1$ and $\nu = \gamma$ respectively, while $\mu = 0$ and $\nu = 1 = \alpha$ in the case when $\gamma = 0$.

Next we introduce two auxiliary functions. Let

$$\phi_{\mu,\nu}(z) = 1 + \sum_{n=1}^{\infty} \frac{(n\nu+1)(n\mu+1)}{n+1} z^n, \tag{4}$$

and

$$\begin{aligned} \psi_{\mu,\nu}(z) &= \phi_{\mu,\nu}^{-1}(z) = 1 + \sum_{n=1}^{\infty} \frac{n+1}{(n\nu+1)(n\mu+1)} z^n \\ &= \int_0^1 \int_0^1 \frac{dsdt}{(1-t^\nu s^\mu z)^2}. \end{aligned} \tag{5}$$

Here $\phi_{\mu,\nu}^{-1}$ denotes the convolution inverse of $\phi_{\mu,\nu}$ such that $\phi_{\mu,\nu} * \phi_{\mu,\nu}^{-1} = z/(1-z)$. If $\gamma = 0$, then $\mu = 0$, $\nu = \alpha$, and it is clear that

$$\psi_{0,\alpha}(z) = 1 + \sum_{n=1}^{\infty} \frac{n+1}{n\alpha+1} z^n = \int_0^1 \frac{dt}{(1-t^\alpha z)^2}.$$

If $\gamma > 0$, then $\nu > 0$, $\mu > 0$, and making the change of variables $u = t^\nu$, $v = s^\mu$ results in

$$\psi_{\mu,\nu}(z) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{u^{1/\nu-1} v^{1/\mu-1}}{(1-uvz)^2} dudv.$$

Thus the function $\psi_{\mu,\nu}$ can be written as

$$\psi_{\mu,\nu}(z) = \begin{cases} \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{u^{1/\nu-1} v^{1/\mu-1}}{(1-uvz)^2} dudv, & \gamma > 0; \\ \int_0^1 \frac{dt}{(1-t^\alpha z)^2}, & \gamma = 0,\ \alpha > 0. \end{cases} \tag{6}$$

Let q be the solution of the initial value problem

$$\frac{d}{dt}\left(t^{1/\nu} q(t)\right) = \begin{cases} \frac{1}{\mu\nu} t^{1/\nu-1} \int_0^1 \frac{(1-\delta)-(1+\delta)st}{(1-\delta)(1+st)^3} s^{1/\mu-1} ds, & \gamma > 0, \\ \frac{1}{\alpha} \frac{(1-\delta)-(1+\delta)t}{(1-\delta)(1+t)^3} t^{1/\alpha-1}, & \gamma = 0,\ \alpha > 0, \end{cases} \tag{7}$$

satisfying $q(0) = 1$.

Solving the differential equation (7), we have

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1-\delta)-(1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu-1} w^{1/\nu-1} dsdw. \tag{8}$$

In particular,

$$q_\alpha(t) = \frac{1}{\alpha}\int_0^1 \frac{(1-\delta)-(1+\delta)st}{(1-\delta)(1+st)^3} s^{1/\alpha-1}ds, \; \gamma = 0, \; \alpha > 0. \tag{9}$$

Further let

$$\Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\nu}}dx, \quad \nu > 0, \tag{10}$$

and

$$\Pi_{\mu,\nu}(t) = \begin{cases} \int_t^1 \Lambda_\nu(x) x^{1/\nu-1-1/\mu}dx, & \gamma > 0, \\ \Lambda_\alpha(t), & \gamma = 0, \; (\mu = 0, \; \nu = \alpha > 0). \end{cases} \tag{11}$$

For the function $\Pi_{\mu,\nu}(t)$, we define

$$\mathfrak{M}_{\Pi_{\mu,\nu}}(h_\delta) = \begin{cases} \Re \int_0^1 t^{1/\mu-1}\Pi_{\mu,\nu}(t)\left[h'_\delta(tz) - \frac{(1-\delta)-(1+\delta)t}{(1-\delta)(1+t)^3}\right]dt, & \gamma > 0, \\ \Re \int_0^1 t^{1/\alpha-1}\Pi_{0,\alpha}(t)\left[h'_\delta(tz) - \frac{(1-\delta)-(1+\delta)t}{(1-\delta)(1+t)^3}\right]dt, & \gamma = 0, \end{cases} \tag{12}$$

where $h_\delta(z)$ is defined as

$$h_\delta(z) = \frac{z\left(1 + \frac{\epsilon+2\delta-1}{2-2\delta}z\right)}{(1-z)^2}, \quad |\epsilon| = 1. \tag{13}$$

3. Main Results

Theorem 2. *Let $\mu \geq 0$, $\nu \geq 0$ satisfy* (3). *Define $\beta < 1$ by*

$$\frac{\beta - 1/2}{(1-\beta)} = -\int_0^1 \lambda(t)q(t)dt, \tag{14}$$

where $q(t)$ is the solution of the initial-value problem (7). *Further for $\Lambda_\nu(t)$ and $\Pi_{\mu,\nu}(t)$ defined by* (10) *and* (11) *respectively, assume that $t^{1/\nu}\Lambda_\nu(t) \to 0$, and $t^{1/\nu}\Pi_{\mu,\nu}(t) \to 0$ as $t \to 0^+$. Then for $\delta \in [0, 1/2]$, $V_\lambda(\mathcal{W}_\beta(\alpha,\gamma)) \subset K(\delta)$ if and only if $\mathfrak{M}_{\Pi_{\mu,\nu}}(h_\delta) \geq 0$, where $\mathfrak{M}_{\Pi_{\mu,\nu}}(h_\delta)$ and h_δ are defined by equations* (12) *and* (13) *respectively.*

Proof. As the case $\gamma = 0$ ($\mu = 0$, $\nu = \alpha$) corresponds to the Theorem 2.3 in [5], so we will prove the result only when $\gamma > 0$.
Let

$$H(z) = (1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma)f'(z) + \gamma z f''(z).$$

Since $\mu+\nu=\alpha-\gamma$ and $\mu\nu=\gamma$, therefore

$$\begin{aligned} H(z) &= (1+\gamma-(\alpha-\gamma))\frac{f(z)}{z}+(\alpha-\gamma-\gamma)f'(z)+\gamma z f''(z) \\ &= (1+\mu\nu-\mu-\nu)\frac{f(z)}{z}+(\mu+\nu-\mu\nu)f'(z)+\mu\nu z f''(z). \end{aligned}$$

Writing $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$, we obtain from (4)

$$H(z)=1+\sum_{n=1}^{\infty}a_{n+1}(n\nu+1)(n\mu+1)z^n=f'(z)*\phi_{\mu,\nu}(z),$$

and (5) gives that

$$f'(z)=H(z)*\psi_{\mu,\nu}(z). \tag{15}$$

Now, for $f\in\mathcal{W}_\beta(\alpha,\gamma)$, we have

$$\Re\left\{e^{i\phi}\frac{H(z)-\beta}{1-\beta}\right\}>0.$$

Thus, in the view of the Theorem 1, we may confine ourselves to functions $f\in\mathcal{W}_\beta(\alpha,\gamma)$ for which

$$H(z)=\beta+(1-\beta)\left(\frac{1+xz}{1+yz}\right),\ \ |x|=|y|=1.$$

Thus (15) gives

$$f'(z)=\left((1-\beta)\frac{1+xz}{1+yz}+\beta\right)*\psi_{\mu,\nu}(z),$$

and therefore

$$\frac{f(z)}{z}=\frac{1}{z}\int_0^z\left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)dw*\psi(z). \tag{16}$$

Here $\psi:=\psi_{\mu,\nu}$.

A well-known result from the theory of convolutions [9, Pg 94] (also see [11]) states that

$$F\in K(\delta)\ \ \Leftrightarrow\ \ \frac{1}{z}(zF'*h_\delta)(z)\neq 0,\ \ z\in E,$$

where h_δ is as defined in (13). Hence $F\in K(\delta)$ if and only if

$$0\neq\frac{1}{z}(V_\lambda(f)(z)*z{h'}_\delta(z))=\frac{1}{z}\left[\int_0^1\lambda(t)\frac{f(tz)}{t}dt*z{h'}_\delta(z)\right]=\int_0^1\frac{\lambda(t)}{1-tz}dt*\frac{f(z)}{z}*{h'}_\delta(z)$$

Using (16), we have

$$\begin{aligned}
0 &\neq \int_0^1 \frac{\lambda(t)}{1-tz}dt * \left[\frac{1}{z}\int_0^z \left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)dw * \psi(z)\right] * h'_\delta(z) \\
&= \int_0^1 \frac{\lambda(t)}{1-tz}dt * h'_\delta(z) * \left[\frac{1}{z}\int_0^z \left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)dw\right] * \psi(z) \\
&= \int_0^1 \lambda(t)h'_\delta(tz)dt * (1-\beta)\left[\frac{1}{z}\int_0^z \left(\frac{1+xw}{1+yw}+\frac{\beta}{(1-\beta)}\right)dw\right] * \psi(z) \\
&= (1-\beta)\left[\int_0^1 \lambda(t)h'_\delta(tz)dt + \frac{\beta}{(1-\beta)}\right] * \frac{1}{z}\int_0^z \frac{1+xw}{1+yw}dw * \psi(z) \\
&= (1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'_\delta(tw)dw\right)dt + \frac{\beta}{(1-\beta)}\right] * \frac{1+xz}{1+yz} * \psi(z).
\end{aligned}$$

This holds if and only if [11, p. 23]

$$\begin{aligned}
&\Re(1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'_\delta(tw)dw\right)dt + \frac{\beta}{(1-\beta)}\right] * \psi(z) \geq 1/2, \\
\Leftrightarrow\quad &\Re(1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'_\delta(tw)dw\right)dt + \frac{\beta}{(1-\beta)} - \frac{1}{2(1-\beta)}\right] * \psi(z) \geq 0, \\
\Leftrightarrow\quad &\Re\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'_\delta(tw)dw\right)dt + \frac{\beta-1/2}{(1-\beta)}\right] * \psi(z) \geq 0, \\
\Leftrightarrow\quad &\Re\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'_\delta(tw)dw - q(t)\right)dt\right] * \psi(z) \geq 0, \quad \text{(using (14))}, \\
\Leftrightarrow\quad &\Re\left[\int_0^1 \lambda(t)\left(h'_\delta(tz) - q(t)\right)dt\right] * \frac{1}{z}\int_0^z \psi(w)dw \geq 0, \\
\Leftrightarrow\quad &\Re\left[\int_0^1 \lambda(t)\left(h'_\delta(tz) - q(t)\right)dt\right] * \sum_{n=0}^{\infty} \frac{z^n}{(n\nu+1)(n\mu+1)} \geq 0, \quad \text{(using (5))} \\
\Leftrightarrow\quad &\Re\int_0^1 \lambda(t)\left(\sum_{n=0}^{\infty} \frac{z^n}{(n\nu+1)(n\mu+1)} * h'_\delta(tz) - q(t)\right)dt \geq 0, \\
\Leftrightarrow\quad &\Re\int_0^1 \lambda(t)\left(\int_0^1\int_0^1 \frac{d\eta d\zeta}{(1-\eta^\nu\zeta^\mu z)} * h'_\delta(tz) - q(t)\right)dt \geq 0, \\
\Leftrightarrow\quad &\Re\int_0^1 \lambda(t)\left(\int_0^1\int_0^1 h'_\delta(tz\eta^\nu\zeta^\mu)d\eta d\zeta - q(t)\right)dt \geq 0,
\end{aligned}$$

which can also be written as

$$\Re\int_0^1 \lambda(t)\left(\int_0^1\int_0^1 \frac{1}{\mu\nu}h'_\delta(tzuv)u^{1/\nu-1}v^{1/\mu-1}dvdu - q(t)\right)dt \geq 0.$$

Writing $w = tu$, we get

$$\Re \int_0^1 \frac{\lambda(t)}{t^{1/\nu}} \left[\int_0^t \int_0^1 h'_\delta(wzv) w^{1/\nu-1} v^{1/\mu-1} dv dw - \mu\nu t^{1/\nu} q(t) \right] dt \geq 0.$$

An integration by parts with respect to t and (7) gives

$$\Re \int_0^1 \Lambda_\nu(t) \left[\int_0^1 h'_\delta(tzv) t^{1/\nu-1} v^{1/\mu-1} dv - t^{1/\nu-1} \int_0^1 \frac{1-\delta-(1+\delta)st}{(1-\delta)(1+st)^3} s^{1/\mu-1} ds \right] dt \geq 0.$$

Again writing $w = vt$ and $\eta = st$ above inequality reduces to

$$\Re \int_0^1 \Lambda_\nu(t) t^{1/\nu-1/\mu-1} \left[\int_0^t h'_\delta(wz) w^{1/\mu-1} dw - \int_0^t \frac{1-\delta-(1+\delta)\eta}{(1-\delta)(1+\eta)^3} \eta^{1/\mu-1} d\eta \right] dt \geq 0,$$

which after integration by parts with respect to t yields

$$\Re \int_0^1 \Pi_{\mu,\nu}(t) t^{1/\mu-1} \left[h'_\delta(tz) - \frac{1-\delta-(1+\delta)t}{(1-\delta)(1+t)^3} \right] dt \geq 0.$$

Thus $F \in K(\delta)$ if and only if $\mathfrak{M}_{\Pi_{\mu,\nu}}(h_\delta) \geq 0$.

Finally, to prove the sharpness, let $f \in \mathcal{W}_\beta(\alpha, \gamma)$ be of the form for which

$$(1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) = \beta + (1-\beta)\frac{1+z}{1-z}.$$

Using a series expansion we obtain that

$$f(z) = z + 2(1-\beta) \sum_{n=1}^\infty \frac{1}{(n\nu+1)(n\mu+1)} z^{n+1}.$$

Thus

$$F(z) = V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt = z + 2(1-\beta) \sum_{n=1}^\infty \frac{\tau_n}{(n\nu+1)(n\mu+1)} z^{n+1},$$

where $\tau_n = \int_0^1 \lambda(t) t^n dt$. From (7), it is a simple exercise to write $q(t)$ in a series expansion as

$$q(t) = 1 + \frac{1}{1-\delta} \sum_{n=1}^\infty \frac{(-1)^n (n+1)(n+1-\delta)}{(n\nu+1)(n\mu+1)} t^n. \qquad (17)$$

Now, by (14) and (17), we have

$$\begin{aligned}
\frac{\beta-1/2}{1-\beta} &= -\int_0^1 \lambda(t)q(t)dt \\
&= -\int_0^1 \lambda(t)\left[1+\frac{1}{1-\delta}\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)(n+1-\delta)}{(n\nu+1)(n\mu+1)}t^n\right]dt \\
&= -1-\frac{1}{1-\delta}\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)(n+1-\delta)}{(n\nu+1)(n\mu+1)}\int_0^1 \lambda(t)t^n dt.
\end{aligned}$$

Therefore

$$\frac{1}{2(1-\beta)} = -\frac{1}{1-\delta}\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)(n+1-\delta)\tau_n}{(n\nu+1)(n\mu+1)}. \tag{18}$$

Finally, we see that

$$F'(z) = 1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(n+1)\tau_n}{(n\nu+1)(n\mu+1)}z^n.$$

Therefore

$$(zF'(z))' = 1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(n+1)^2\tau_n}{(n\nu+1)(n\mu+1)}z^n.$$

For $z=-1$, we have

$$\begin{aligned}
(zF')'(-1) &= 1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)^2\tau_n}{(n\nu+1)(n\mu+1)} \\
&= 1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)(n+1-\delta)\tau_n}{(n\nu+1)(n\mu+1)}+2(1-\beta)\sum_{n=1}^{\infty}\frac{(-1)^n\delta(n+1)\tau_n}{(n\nu+1)(n\mu+1)} \\
&= 1-(1-\delta)+\delta 2(1-\beta)\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)\tau_n}{(n\nu+1)(n\mu+1)} \quad \text{(Using(18))} \\
&= \delta\left(1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)\tau_n}{(n\nu+1)(n\mu+1)}\right) \\
&= \delta F'(-1).
\end{aligned}$$

Thus $(zF'(z))'/F'(z)$ at $z=-1$ equals δ. This implies that the result is sharp for the order of convexity.

4. Consequences of Theorem 2

To obtain a sufficient condition for the convexity of order δ of the integral transform (1) by a much easier method, we present the following theorem.

Theorem 3. *Let $\Lambda_\nu(t)$, $\Pi_{\mu,\nu}(t)$ be integrable on [0,1] and positive on (0,1). Also, suppose that $t^{1/\nu}\Lambda_\nu(t) \to 0$, and $t^{1/\nu}\Pi_{\mu,\nu}(t) \to 0$ as $t \to 0^+$. Assume further that $\mu \geq 1$ and*

$$\frac{\left(-t\Pi'_{\mu,\nu}(t) + \left(1-\frac{1}{\mu}\right)\Pi_{\mu,\nu}(t)\right)}{(1+t)(1-t)^{1+2\delta}} \text{ is decreasing on (0,1).} \tag{19}$$

For $\delta \in [0, 1/2]$, if β satisfies (14), then $V_\lambda(f) \in K(\delta)$ for $f \in \mathcal{W}_\beta(\alpha,\gamma)$.

Proof. For $\gamma > 0$, integration by parts with respect to t yields

$$\begin{aligned}
& \int_0^1 t^{\frac{1}{\mu}-1}\Pi_{\mu,\nu}(t)\left(\Re(h'_\delta(tz)) - \frac{1-\delta-(1+\delta)t}{(1-\delta)(1+t)^3}\right)dt \\
= & \int_0^1 t^{\frac{1}{\mu}-1}\Pi_{\mu,\nu}(t)\frac{d}{dt}\left(\Re\frac{h_\delta(tz)}{z} - \frac{t(1-\delta(1+t))}{(1-\delta)(1+t)^2}\right)dt \\
= & \int_0^1 t^{\frac{1}{\mu}-1}\left(-t\Pi'_{\mu,\nu}(t) + \left(1-\frac{1}{\mu}\right)\Pi_{\mu,\nu}(t)\right)\left(\Re\frac{h_\delta(tz)}{tz} - \frac{1-\delta(1+t)}{(1-\delta)(1+t)^2}\right)dt.
\end{aligned}$$

Also for $\mu \geq 1$, the function $t^{1/\mu-1}$ is decreasing on (0,1). Thus, the condition (19) along with Theorem 1 from [8] yields

$$\int_0^1 t^{\frac{1}{\mu}-1}\Pi_{\mu,\nu}(t)\left(\Re\big(h'_\delta(tz)\big) - \frac{1-\delta-(1+\delta)t}{(1-\delta)(1+t)^3}\right)dt > 0.$$

Thus, an application of Theorem 2 evidently leads to the desired result.

Below, we obtain the conditions to ensure convexity of $V_\lambda(f)$. As defined in (11) and (10), for $\gamma > 0$,

$$\Pi_{\mu,\nu}(t) = \int_t^1 \Lambda_\nu(x)x^{1/\nu-1-1/\mu}dx, \text{ and } \Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\nu}}dx.$$

In order to apply Theorem 3, we have to prove that the function

$$k(t) = \frac{\left(t^{\frac{1}{\nu}-\frac{1}{\mu}}\Lambda_\nu(t) + \left(1-\frac{1}{\mu}\right)\Pi_{\mu,\nu}(t)\right)}{(1+t)(1-t)^{1+2\delta}} := \frac{p(t)}{(1+t)(1-t)^{1+2\delta}}$$

is decreasing in (0,1). Since $k(t) > 0$ and

$$\begin{aligned}\frac{k'(t)}{k(t)} &= \frac{p'(t)}{p(t)} + \frac{2(t+\delta(1+t))}{1-t^2} \\ &= \frac{2(t+\delta(1+t))}{(1-t^2)p(t)}\left[\frac{(1-t^2)p'(t)}{2(t+\delta(1+t))} + p(t)\right] = \frac{2(t+\delta(1+t))}{(1-t^2)p(t)}\left[q(t)\right] \quad \text{(say).}\end{aligned}$$

Thus to prove that $k'(t) \leq 0$, it is enough to prove that $q(t) \leq 0$. Since $q(1) = 0$, so it remains to show that $q(t)$ is increasing over (0,1). Now

$$q'(t) = \frac{(1+t)}{2(t+\delta(1+t))^2}\left[(1-t)(t+\delta(1+t))p''(t) - (1-t-\delta(1+t))(1+2\delta)p'(t)\right].$$

So, $q'(t) \geq 0$ for $t \in (0,1)$ is equivalent to the inequality $r(t) \geq 0$, where

$$r(t) = (1-t)(t+\delta(1+t))p''(t) - (1-t-\delta(1+t))(1+2\delta)p'(t)$$

By using the idea similar to the one used to prove Theorem 3.1 in [6], we can write

$$r(t) = -\lambda(t)t^{1-\frac{1}{\mu}}\left[\left(\frac{1}{\nu} - \frac{1}{\mu} - 1\right)X(t) + Z(t) + \frac{t\lambda'(t)}{\lambda(t)}X(t)\right] + \left[\left(\frac{1}{\nu} - \frac{1}{\mu} - 1\right)X(t) + Z(t)\right]\left(\frac{1}{\nu} - 1\right)t^{\frac{1}{\nu}-\frac{1}{\mu}-1}\int_t^1 A(s)ds$$

where,

$$\begin{aligned} A(t) &= \lambda(t)t^{-1/\nu}, \\ X(t) &= (1-t)(t+\delta(1+t)), \\ Z(t) &= -t(1-t-\delta(1+t))(1+2\delta). \end{aligned} \tag{20}$$

Clearly, $A(t) > 0$ and $X(t) > 0$ for all $t \in (0,1)$.
Thus, $r(t)$ is non-negative if

$$\left(\frac{1}{\nu} - \frac{1}{\mu} - 1\right)X(t) + Z(t) + \frac{t\lambda'(t)}{\lambda(t)}X(t) \leq 0 \ \text{ and } \ \left[\left(\frac{1}{\nu} - \frac{1}{\mu} - 1\right)X(t) + Z(t)\right]\left(\frac{1}{\nu} - 1\right) \geq 0. \tag{21}$$

Since $\nu \geq 1$, we can rewrite the condition (21) as follows :

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu} - \left(\frac{X(t)+Z(t)}{X(t)}\right) \ \text{ and } \ \frac{1}{\nu} - \frac{1}{\mu} - 2 \leq -\left(\frac{X(t)+Z(t)}{X(t)}\right).$$

In view of the fact that $X(t) + Z(t)$ and $X(t)$ are non-negative on (0,1), the above inequality further reduces to

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu} \ \text{ and } \ \frac{1}{\nu} - \frac{1}{\mu} - 2 \leq 0. \tag{22}$$

For $\mu \geq 1$, condition (3) implies $\nu \geq \mu \geq 1$. Thus, condition (22) implies that $r(t)$ is non-negative if

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \qquad \nu \geq \mu \geq 1. \tag{23}$$

These conditions leads to the following theorem.

Theorem 4. *Assume that both $\Lambda_\nu(t)$, $\Pi_{\mu,\nu}(t)$ are integrable on [0,1] and positive on (0,1). Let $\lambda(t)$ be a non-negative real-valued integrable function on [0,1] and satisfy the condition*

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \qquad \nu \geq \mu \geq 1.$$

Let $f \in \mathcal{W}_\beta(\alpha, \gamma)$ and $\beta < 1$ with

$$\frac{\beta - 1/2}{(1-\beta)} = -\int_0^1 \lambda(t) q(t) dt,$$

where $q(t)$ is defined by (8). Then $F(z) = V_\lambda(f)(z) \in K(\delta)$ for $\delta \in [0, 1/2]$. The conclusion does not hold for smaller values of β.

On the other hand, when $\gamma = 0$ ($\mu = 0$, $\nu = \alpha > 0$), so we get the following result.

Theorem 5. *Let $\lambda(t)$ be a non-negative real-valued integrable function on [0,1]. Assume that both $\Lambda_\alpha(t)$, $\Pi_{0,\alpha}(t)$ are integrable on [0,1] and positive on (0,1). Let $\lambda(1) = 0$ and λ satisfies the condition*

$$t\lambda''(t) - \frac{1}{\alpha}\lambda'(t) \geq 0, \qquad \alpha \geq 1.$$

Let $f \in \mathcal{W}_\beta(\alpha, \gamma)$ and $\beta < 1$ with

$$\frac{\beta - 1/2}{(1-\beta)} = -\int_0^1 \lambda(t) q_\alpha(t) dt,$$

where $q_\alpha(t)$ is defined by 9 with $\delta \in [0, 1/2]$. Then $F(z) = V_\lambda(f)(z) \in K(\delta)$. The conclusion does not hold for smaller values of β.

Proof. As in Theorem 2, for $\gamma = 0$ and $f \in \mathcal{W}_\beta(\alpha, \gamma)$, we have $V_\lambda(f)(z) \in K(\delta)$ if

$$\int_0^1 t^{\frac{1}{\alpha}-1} \Pi_{0,\alpha}(t) \left(\Re\big(h'_\delta(tz)\big) - \frac{1 - \delta - (1+\delta)t}{(1-\delta)(1+t)^3} \right) dt > 0,$$

which is equivalent to

$$\int_0^1 t^{\frac{1}{\alpha}-1} \left(t^{1-\frac{1}{\alpha}} \lambda(t) + \left(1 - \frac{1}{\alpha}\right) \Lambda_\alpha(t) \right) \left(\Re \frac{h_\delta(tz)}{tz} - \frac{1 - \delta(1+t)}{(1-\delta)(1+t)^2} \right) dt > 0.$$

Since $t^{\frac{1}{\alpha}-1}$ is decreasing on (0,1) for $\alpha \geq 1$, thus to apply Theorem 1 in [8], it is enough to show that

$$p(t) = \frac{t^{1-\frac{1}{\alpha}}\lambda(t) + \left(1-\frac{1}{\alpha}\right)\Lambda_\alpha(t)}{(1+t)(1-t)^{1+2\delta}} := \frac{k(t)}{(1+t)(1-t)^{1+2\delta}}$$

is decreasing on (0,1). Here, logarithmic differentiation implies that

$$\frac{p'(t)}{p(t)} = \frac{2(t+\delta(1+t))}{(1-t^2)k(t)}\left[\frac{(1-t^2)k'(t)}{2(t+\delta(1+t))} + k(t)\right].$$

Since $p(t) > 0$ for $\alpha \geq 1$, thus to prove that $p'(t) \leq 0$ on (0,1) it remains to show that

$$r(t) = k(t) + \frac{(1-t^2)k'(t)}{2(t+\delta(1+t))} \leq 0.$$

Since $r(1) = 0$, so $r(t) \leq 0$ if $r(t)$ is increasing on (0,1). Thus, $r'(t)$ is non-negative if

$$\frac{t^{\frac{-1}{\alpha}}(1+t)}{2(t+\delta(1+t))}\left\{X(t)t\lambda''(t) + \left[\left(1-\frac{1}{\alpha}\right)X(t) + Z(t)\right]\lambda'(t)\right\} \geq 0,$$

where $X(t)$ and $Z(t)$ are as defined in (20). Further simplification yields that

$$t\lambda''(t) + \left(\frac{X(t)+Z(t)}{X(t)} - \frac{1}{\alpha}\right)\lambda'(t) \geq 0.$$

Since, $X(t)$ and $X(t)+Z(t)$ are non-negative in (0,1), thus $r'(t) \geq 0$ is equivalent to

$$t\lambda''(t) - \frac{1}{\alpha}\lambda'(t) \geq 0, \qquad \alpha \geq 1,$$

which completes the proof.

Remark 1. *Observe that results in [2] can be obtained from our results by setting* $\delta = 0$.

5. Applications

In this section, we apply Theorem 4 and Theorem 5 to obtain certain results regarding convexity of well-known integral operators. The proofs of the following results run on the same lines as given in [2] and hence omitted.

Consider λ to be defined as

$$\lambda(t) = (1+c)t^c, \qquad c > -1.$$

Then the integral transform

$$F_c(z) = V_\lambda(f)(z) = (1+c)\int_0^1 t^{c-1} f(tz)dt, \qquad c > -1, \tag{24}$$

is the well-known Bernardi integral operator. The classical Alexander and Libera transforms are special cases of (24) with $c = 0$ and $c = 1$ respectively. For this special case of λ, the following result holds.

Theorem 6. *Let $c > -1$ and $0 < \gamma \le \alpha \le 1 + 2\gamma$. Let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1-\beta} = -(1+c)\int_0^1 t^c q(t)dt,$$

where q is given by

$$q(t) = \frac{1}{\mu\nu}\int_0^1\int_0^1 \frac{(1-\delta)-(1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu-1} w^{1/\nu-1} ds dw.$$

Then for $\delta \in [0, 1/2]$, we have $V_\lambda(\mathcal{W}_\beta(\alpha,\gamma)) \subset K(\delta)$ provided c satisfies the condition:

$$c \le 2 + \frac{1}{\mu} - \frac{1}{\nu}, \;\; \nu \ge \mu \ge 1.$$

The value of β is sharp.

Writing $\alpha = 1 + 2\gamma$, $\gamma > 0$ and $\mu = 1$ in Theorem 6 gives the following criteria of convexity:

Corollary 7. *Let $-1 < c \le 3 - 1/\gamma$ and $\gamma \ge 1$. Let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1-\beta} = -(1+c)\int_0^1 t^c q(t)dt,$$

where q is given by

$$q(t) = \frac{1}{\mu\nu}\int_0^1\int_0^1 \frac{(1-\delta)-(1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu-1} w^{1/\nu-1} ds dw.$$

Then for $\delta \in [0, 1/2]$, we have $V_\lambda(\mathcal{R}_\beta(\gamma)) \subset K(\delta)$. The value of β is sharp.

Further, letting $\gamma = 1$ and $c = 0$ in Corollary 7, we have

Corollary 8. *Let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1-\beta} = \frac{1}{1-\delta}\left(\delta\frac{\pi^2}{12} - \log 2\right)$$

If $f \in \mathcal{R}_\beta(1)$, then Alexander transform $F_0(z) \equiv A[f](z) = \displaystyle\int_0^1 \frac{f(tz)}{t}dt$ is convex of order δ where $\delta \in [0, 1/2]$. The value of β is sharp.

Remark 2. *1. For $\delta = 0$,*

$$\beta_0 = \frac{1 - 2\log 2}{2 - 2\log 2} = -0.629\ldots.$$

Then, for f satisfying

$$\Re e^{i\phi}\left(f'(z) + zf''(z) - \beta\right) > 0,\ z \in E,$$

Alexander transform $A[f]$ is convex. It has been shown in [8] that β_0 is the best possible bound here.

2. *We note that for $\delta = 1/2$, $\beta_{1/2} = 0.590\ldots$. Then, for f satisfying*

$$\Re e^{i\phi}\left(f'(z) + zf''(z) - \beta\right) > 0,\ z \in E,$$

Alexander transform $A[f]$ is convex of order $1/2$.

While, the case $c = 0$ in Theorem 6 yields yet another interesting result, which we state as a theorem.

Theorem 9. *Let $0 < \gamma \le \alpha \le 1 + 2\gamma$. If $F \in \mathcal{A}$ satisfies*

$$\Re\left(F'(z) + \alpha zF''(z) + \gamma z^2 F'''(z)\right) > \beta, \qquad z \in E,$$

and $\beta < 1$ satisfies

$$\frac{\beta - 1/2}{1 - \beta} = -\int_0^1 q(t)dt,$$

where q is given by

$$q(t) = \frac{1}{\mu\nu}\int_0^1\int_0^1 \frac{(1-\delta) - (1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu - 1} w^{1/\nu - 1} ds dw,$$

then for $\delta \in [0, 1/2]$, F belongs to $K(\delta)$. The value of β is sharp.

To state our next theorem, we define

$$\lambda(t) = \begin{cases} (a+1)(b+1)\frac{t^a(1-t^{b-a})}{b-a}, & \text{b}\neq\text{a};\\ (a+1)^2 t^a \log(1/t), & \text{b=a}, \end{cases} \tag{25}$$

where $b > -1$ and $a > -1$.
Then,

$$V_\lambda(f)(z) = G_f(a, b; z) = \begin{cases} \frac{(a+1)(b+1)}{b-a}\int_0^1 t^{a-1}(1-t^{b-a})f(tz)dt, & \text{b}\neq\text{a};\\ (a+1)^2\int_0^1 t^{a-1}\log(1/t)f(tz)dt, & \text{b=a}. \end{cases}$$

Theorem 10. *Let $b > -1$, $a > -1$ and $0 < \gamma \leq \alpha \leq 1 + 2\gamma$. Let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1 - \beta} = -\int_0^1 \lambda(t) q(t) dt,$$

where q is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1-\delta) - (1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu - 1} w^{1/\nu - 1} ds dw.$$

and $\lambda(t)$ is defined by (25). If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then the convolution operator $G_f(a, b; z)$ belongs to $K(\delta)$ with $\delta \in [0, 1/2]$ if

$$a \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \qquad\qquad \nu \geq \mu \geq 1.$$

The value of β is sharp.

Substituting $\alpha = 1 + 2\gamma$, $\gamma > 0$ and $\mu = 1$ in Theorem 6, gives the following result :

Corollary 11. *Let $b > -1$, $-1 < a \leq 3 - 1/\gamma$ and $\gamma \geq 1$. Let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1 - \beta} = -\int_0^1 \lambda(t) q(t) dt,$$

where q is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1-\delta) - (1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu - 1} w^{1/\nu - 1} ds dw.$$

and $\lambda(t)$ is defined by (25). If $f \in \mathcal{R}_\beta(\gamma)$, then the convolution operator $G_f(a, b; z)$ belongs to $K(\delta)$ with $\delta \in [0, 1/2]$. The value of β is sharp.

While for $\gamma = 0$, with an application of Theorem 5, we get the following result:

Theorem 12. *Let $b > -1$, $a > -1$ and $\alpha \geq 1$. Let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1 - \beta} = -\int_0^1 \lambda(t) q_\alpha(t) dt,$$

where q_α is given by

$$q_\alpha(t) = \frac{1}{\alpha} \int_0^1 \frac{(1-\delta) - (1+\delta)st}{(1-\delta)(1+st)^3} s^{1/\alpha - 1} ds$$

and $\lambda(t)$ is defined by (25). If $f \in \mathcal{P}_\beta(\alpha)$, then the convolution operator $G_f(a, b; z)$ belongs to $K(\delta)$ with $\delta \in [0, 1/2]$ if one of the following conditions holds :
(i) $-1 < a \leq 0$ and $a = b$, or
(ii) $-1 < a \leq 0$ and $-1 < a < b \leq 1 + 1/\alpha$.
The value of β is sharp.

Now, we define

$$\lambda(t) = \frac{(1+a)^p}{\Gamma(p)} t^a \left(\log(1/t)\right)^{p-1}, \;\; a > -1, \;\; p \geq 0.$$

In this case, V_λ reduces to the Komatu operator [9]

$$V_\lambda(f)(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 \left(\log\left(\frac{1}{t}\right)\right)^{p-1} t^{a-1} f(tz)dt, \;\; a > -1, \;\; p \geq 0.$$

For $p = 1$ Komatu operator gives the Bernardi integral operator. For this λ, the following result holds.

Theorem 13. *Let $a > p - 2 > -1$ and $0 < \gamma \leq \alpha \leq 1 + 2\gamma$. Let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1 - \beta} = -\int_0^1 \lambda(t) q(t) dt,$$

where q is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{(1-\delta) - (1+\delta)swt}{(1-\delta)(1+swt)^3} s^{1/\mu - 1} w^{1/\nu - 1} ds dw.$$

If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then the function

$$\Phi_p(a; z) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 \left(\log\left(\frac{1}{t}\right)\right)^{p-1} t^{a-1} f(tz) dt$$

belongs to $K(\delta)$ with $\delta \in [0, 1/2]$ if

$$a \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \qquad\qquad \nu \geq \mu \geq 1.$$

The value of β is sharp.

References

[1] R. M. Ali, A. O. Badghaish, V. Ravichandran and A. Swaminathan, *Starlikeness of integral transforms and duality*, J. Math. Anal. Appl. 385 (2012), 808-822.

[2] R. M. Ali, M. M. Nargesi and V. Ravichandran, *Convexity of integral transforms and duality*, Complex Var. Elliptic Equ. (2012), 1-22.

[3] R. M. Ali and V. Singh, *Convexity and starlikeness of functions defined by a class of integral operators*, Complex Variables Theory Appl. 26, 4 (1995), 299-309.

[4] R. Balasubramanian, S. Ponnusamy and D. J. Prabhakaran, *Duality techniques for certain integral transforms to be starlike*, J. Math. Anal. Appl. 293 (2004), 355-373.

[5] R. Balasubramanian, S. Ponnusamy and D. J. Prabhakaran, *Convexity of integral transforms and function spaces*, Integral Transforms Spec. Funct. 18, 1-2 (2007), 1-14.

[6] R. Balasubramanian, S. Ponnusamy and D. J. Prabhakaran, *On extremal problems related to integral transforms of a class of analytic functions*, J. Math. Anal. Appl. 336, 1 (2007), 542-555.

[7] J. H. Choi, Y.C. Kim and M. Saigo, *Geometric properties of convolution operators defined by Gaussian hypergeometric functions*, Integral Transforms Spec. Funct. 13, 2 (2002), 117-130.

[8] R. Fournier and S. Ruscheweyh, *On two extremal problems related to univalent functions*, Rocky Mountain J. Math. 24, 2 (1994), 529-538.

[9] Y. Komatu, *On analytic prolongation of a family of operators*, Mathematica (Cluj) 32, 55(2) (1990), 141-145.

[10] S. Ruscheweyh, *Convolutions in Geometric Function Theory (Montreal: Les Presses de l′Universite de Montreal).*

[11] S. Ruscheweyh, *Duality for Hadamard products with applications to extremal problems for functions regular in unit disc*, Trans. Amer. Math. Soc. 210 (1975), 63-74.

Sarika Verma, Sushma Gupta and Sukhjit Singh
Department of Mathematics,
Sant Longowal Institute of Engineering and Technology,
Longowal-148106, (Punjab) India
email: *sarika.16984@gmail.com*