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FIXED POINT THEOREM FOR N METRIC SPACES

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Abstract. Various fixed point theorem on two, three, four and more metric spaces have been proved by many authors. In this paper, we prove a fixed point theorem on n-metric spaces for set valued mapping. Our theorem generalizes and extends many previously proved results .

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1. Introduction and Preliminaries

Fixed point theory for set valued and associated mappings has played an essential role in many aspects of nonlinear functional analysis. Because of its simplicity, still the Banach fixed point theorem seems to be the most important result in metric fixed point theory. The first considerable generalization of Banach's theorem was obtained by E. Rakotch in 1962, who replaced Banach's strict contractions with contractive mappings. The fixed point theory of set-valued contractions, which was initiated by S.B. Nadler [9]. After that it was developed in different directions by many authors, in particular, by S.Reich [10] and B. Fisher [2]. Such mappings, as well as their many modifications, were studied and used by many authors.
The following fixed point theorem was proved by R.K. Jain *et al* [5]:

Theorem 1. *Let (X, d), (Y, ρ) and $Z, \sigma)$ be complete metric spaces. If T is a continuous mapping of X into Y, S is a continuous mapping of Y into Z and R is a mapping of Z into X.*

$$\begin{aligned} d\left(RSTx, RSTx'\right) &\leqslant c \max\left\{d\left(x, x'\right), d\left(x, RSTx\right),\right. \\ &\quad \left. d\left(x', RSTx'\right), \rho\left(Tx, Tx'\right), \sigma\left(STX, STx'\right)\right\} \\ \rho\left(TRSy, TRSy'\right) &\leqslant c \max\left\{\rho\left(y, y'\right), \rho\left(y, TRSy\right),\right. \\ &\quad \left. \rho\left(y', TRSy'\right), \sigma\left(Sy, Sy'\right), d\left(RSy, RSy'\right)\right\} \\ \sigma\left(STRz, STRz'\right) &\leqslant c \max\left\{\sigma\left(z, z'\right), \sigma\left(z, STRz\right),\right. \\ &\quad \left. \sigma\left(z', STRz'\right), d\left(Rz, Rz'\right), \rho\left(TRz, TRz'\right)\right\} \end{aligned}$$

$\forall$ $x, x' \in X, y, y' \in Y$ and $z, z' \in Z$ where $0 \leqslant c < 1$ then RST has a unique fixed point $u \in X$, TRS has a unique fixed point $v \in Y$ and STR has a unique fixed point $w \in Z$. Further $Tu = v$, $Sv = w$ and $Rw = u$.

A number of fixed point theorems for set valued mapping and for n metric complete metric spaces are obtained by many authors, which generalized the result of [2]. Recently, Vishal Gupta, Ashima Kanwar[8] also proved the following related fixed point theorem in n-complete metric spaces.

Theorem 2. *Let (X_i, d_i) be complete metric spaces where $i = 1, 2, 3, \ldots, n$. If A_i is continuous mapping of X_i to X_{i+1} where $i = 1, 2, 3, \ldots, n-1$ and A_n is mapping of X_n to X_1 satisfying the following inequalities.*

$$\begin{aligned} &d_1(A_nA_{n-1}\ldots A_2A_1x^1, A_nA_{n-1}\ldots A_2A_1x_1^1) \\ &\quad\leqslant c\max\left\{d_1\left(x^1, x_1^1\right), d_1\left(x^1, A_nA_{n-1}\ldots A_2A_1x^1\right),\right. \\ &\quad\quad d_1\left(x_1^1, A_nA_{n-1}\ldots A_2A_1x_1^1\right), d_2\left(A_1x^1, A_1x_1^1\right), \\ &\quad\quad d_3\left(A_2A_1x^1, A_2A_1x_1^1\right), d_4\left(A_3A_2A_1x^1, A_3A_2A_1x_1^1\right), \\ &\quad\quad \left.\ldots, d_n\left(A_{n-1}A_{n-2}\ldots A_2A_1x^1, A_{n-1}A_{n-2}\ldots A_2A_1x_1^1\right)\right\} \end{aligned} \tag{1}$$

$$\begin{aligned} &d_2(A_1A_n\ldots A_3A_2x^1, A_1A_n\ldots A_3A_2x_1^2) \\ &\quad\leqslant c\max\left\{d_2\left(x^2, x_1^2\right), d_2\left(x^2, A_1A_n\ldots A_3A_2x^2\right),\right. \\ &\quad\quad d_2\left(x_1^2, A_1A_n\ldots A_3A_2x_1^2\right), d_3\left(A_2x^2, A_2x_1^2\right), \\ &\quad\quad d_4\left(A_3A_2x^2, A_3A_2x_1^2\right), \ldots, \\ &\quad\quad d_n\left(A_{n-1}A_{n-2}\ldots A_3A_2x^2, A_{n-1}A_{n-2}\ldots A_3A_2x_1^2\right), \\ &\quad\quad \left.d_1\left(A_nA_{n-1}\ldots A_3A_2x^2, A_nA_{n-1}\ldots A_3A_2x_1^2\right)\right\} \end{aligned} \tag{2}$$

$$\begin{aligned} &d_3(A_2A_1A_n\ldots A_4A_3x^3, A_2A_1A_n\ldots A_4A_3x_1^3) \\ &\quad\leqslant c\max\left\{d_3\left(x^3, x_1^3\right), d_3\left(x^3, A_2A_1A_n\ldots A_4A_3x^3\right),\right. \\ &\quad\quad d_3\left(x_1^3, A_2A_1A_n\ldots A_4A_3x_1^3\right), d_4\left(A_3x^3, A_3x_1^3\right), \\ &\quad\quad d_5\left(A_4A_3x^3, A_4A_3x_1^3\right), \ldots, \\ &\quad\quad d_n\left(A_{n-1}A_{n-2}\ldots A_4A_3x^3, A_{n-1}A_{n-2}\ldots A_4A_3x_1^3\right), \\ &\quad\quad d_1\left(A_nA_{n-1}\ldots A_4A_3x^3, A_nA_{n-1}\ldots A_4A_3x_1^3\right) \\ &\quad\quad \left.d_2\left(A_1A_n\ldots A_4A_3x^3, A_1A_n\ldots A_4A_3x_1^3\right)\right\} \end{aligned} \tag{3}$$

So continuously like above.

$$\begin{aligned} & d_n(A_{n-1}A_{n-2}\dots A_1A_nx^n, A_{n-1}A_{n-2}\dots A_1A_nx_1^n) \\ & \quad \leqslant c\max\{d_n(x^n, x_1^n), d_n(x^n, A_{n-1}A_{n-2}\dots A_1A_nx^n), \\ & \qquad d_n(x_1^n, A_{n-1}A_{n-2}\dots A_1A_nx_1^n), d_1(A_nx^n, A_nx_1^n), \\ & \qquad d_2(A_1A_nx^n, A_1A_nx_1^n), d_3(A_2A_1A_nx^n, A_2A_1A_nx_1^n), \\ & \qquad \dots, d_{n-1}(A_{n-2}A_{n-3}\dots A_1A_nx^n, A_{n-2}A_{n-3}\dots A_1A_nx_1^n)\} \end{aligned} \tag{4}$$

$$\forall\ x', x_1' \in X_1, x^2, x_1^2 \in X_2, \dots, x^n, x_1^n \in X_n$$

where $0 \leqslant c < 1$. *Then* $A_nA_{n-1}\dots A_2A_1$ *has a unique fixed point* $\beta_1 \in X_1$, $A_1A_n\dots A_3A_2$ *has a unique fixed point* $\beta_2 \in X_2$, $A_2A_1\dots A_4A_3$ *has a unique fixed point*
$\beta_3 \in X_3$ *and so on* $A_{n-1}A_{n-2}\dots A_1A_n$ *has a unique fixed point* $\beta_n \in X_n$. *Further,*

$$A_1(\beta_1) = \beta_2, A_2(\beta_2) = \beta_3, A_3(\beta_3) = \beta_4, \dots, A_{n-1}(\beta_{n-1}) = \beta_n, A_n(\beta_n) = \beta_1.$$

In the next section, we shall prove unique fixed point theorem for set valued mappings in the setting of n mertic spaces and for proving our theorem we need the following definition,

Let (X, d) be a complete metric space and let $B(X)$ be the set of all non-empty subset of X. The function $\delta(A, B)$ with A and B in $B(X)$ is defined by

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$$

and also

$$\delta(a, b) = d(a, b)$$

So

$$\delta(A, B) = \delta(B, A) \geqslant 0$$

$$\delta(A, B) \geqslant \delta(A, C) + \delta(C, B)$$

2. Main Result

Theorem 3. *Let* (X_i, d_i) *be complete metric spaces*
where $i = 1, 2, \dots, n$. *If* $T_1, T_2, \dots, T_{n-1}$ *is continuous mapping such that* $T_i : X_i \mapsto B(X_{i+1})$ *where* $i = 1, 2, \dots, n-1$ *and* $T_n : X_n \mapsto B(X_1)$ *satisfying the following*

inequalities.

$$\begin{aligned}
&\delta_1\left(T_nT_{n-1}\dots T_2T_1x^1, T_nT_{n-1}\dots T_2T_1x^1_{11}\right)\\
&\leqslant c\max\left\{d_1\left(x^1,x^1_{11}\right), \delta_1\left(x^1, T_nT_{n-1}\dots T_2T_1x^1\right),\right.\\
&\quad \delta_1\left(x^1_{11}, T_nT_{n-1}\dots T_2T_1x^1_{11}\right), \delta_2\left(T_1x^1, T_1x^1_{11}\right), \delta_3\left(T_2T_1x^1, T_2T_1x^1_{11}\right), \dots\\
&\quad \left.\delta_n\left(T_{n-1}T_{n-2}\dots T_2T_1x^1, T_{n-1}T_{n-2}\dots T_2T_1x^1_{11}\right)\right\}
\end{aligned} \tag{5}$$

$$\begin{aligned}
\delta_2\left(T_1T_n\dots T_3T_2x^2,\ T_1T_n\dots T_3T_2x^2_{11}\right)
&\leqslant c\max\left\{d_2\left(x^2,x^2_{11}\right),\right.\\
&\quad \delta_2\left(x^2, T_1T_n\dots T_3T_2x^2\right),\\
&\quad \delta_2\left(x^2_{11}, T_1T_n\dots T_3T_2x^2_{11}\right),\\
&\quad \delta_3\left(T_2x^2, T_2x^2_{11}\right), \delta_4\left(T_3T_2x^2, T_3T_2x^2_{11}\right), \dots\\
&\quad \delta_n\left(T_{n-1}T_{n-2}\dots T_3T_2x^2,\right.\\
&\quad \left.T_{n-1}T_{n-2}\dots T_3T_2x^2_{11}\right)\\
&\quad \left.\delta_1\left(T_nT_{n-1}\dots T_2x^2, T_nT_{n-1}\dots T_2x^2_1\right)\right\}
\end{aligned} \tag{6}$$

$$\begin{aligned}
\delta_3\left(T_2T_1T_n\dots T_4T_3x^2, T_2\ T_1T_n\dots T_4T_3x^2_{11}\right)
&\leqslant c\max\left\{d_3\left(x^3,x^3_{11}\right),\right.\\
&\quad \delta_3\left(x^3, T_2T_1T_n\dots T_4T_3x^3\right),\\
&\quad \delta_3\left(x^3_{11}, T_2T_1T_n\dots T_4T_3x^3_{11}\right),\\
&\quad \delta_4\left(T_3x^3, T_3x^3_{11}\right), \delta_5\left(T_4T_3x^2, T_4T_3x^3_{11}\right), \dots\\
&\quad \delta_n\left(T_{n-1}T_{n-2}\dots T_4T_3x^3,\right.\\
&\quad \left.T_{n-1}T_{n-2}\dots T_4T_3x^3_{11}\right)\\
&\quad \delta_1\left(T_nT_{n-1}\dots T_4T_3x^3,\right.\\
&\quad \left.T_nT_{n-1}\dots T_4T_3x^2_1\right),\\
&\quad \delta_2\left(T_1T_n\dots T_4T_3x^3,\right.\\
&\quad \left.\left.T_1T_n\dots T_4T_3x^3_{11}\right)\right\}
\end{aligned} \tag{7}$$

$$\begin{aligned}\delta_n\left(T_{n-1}T_n\ldots T_1T_nx^n,\ T_{n-1}T_{n-2}\ldots T_1T_nx_{11}^n\right)&\\ \leqslant c\max\{d_n\left(x^n,x_{11}^n\right),&\\ \delta_n\left(x^n,T_{n-1}T_{n-2}\ldots T_1T_nx^n\right),&\\ \delta_n\left(x_{11}^n,T_{n-1}T_{n-2}\ldots T_1T_nx_{11}^n\right),&\\ \delta_1\left(T_nx^n,T_nx_{11}^n\right),\delta_2\left(T_1T_2x^n,T_1T_nx_{11}^n\right)&\\ \delta_3\left(T_2T_1T_nx^n,T_2T_1T_nx_{11}^n\right),\ldots&\\ \delta_{n-1}\left(T_{n-2}T_{n-3}\ldots T_1T_nx^n,\right.&\\ \left.T_{n-2}T_{n-3}\ldots T_1T_nx_{11}^n\right)\}&\end{aligned} \tag{8}$$

$\forall\ x^1,x_{11}^1\in X_1,\ x^2,x_{11}^2\in X_2$ and $x^3,x_{11}^3\in X_3$ and so on $x^n,x_{11}^n\in X_n$ where $0\leqslant c<1$ then $T_nT_{n-1}\ldots T_2T_1$ has a unique fixed point $\alpha_1\in X_1,\ T_1T_n\ldots T_3T_2$ has a unique fixed point $\alpha_2\in X_2,\ T_2T_1T_n\ldots T_4T_3$ has a unique fixed point $\alpha_3\in X_3,\ldots,T_{n-1}T_{n-2}\ldots T_1T_2$ has a unique fixed point $\alpha_n\in X_n$. Further $T_1(\alpha_1)=\alpha_2,T_2(\alpha_2)=\alpha_3,T_3(\alpha_3)=\alpha_4,\ldots,T_{n-1}(\alpha_{n-1})=\alpha_n,T_n(\alpha_n)=\alpha_1$.

Proof. Let $x^1=x_1^1$ be an arbitrary point in X_1. We define sequence $\{x_m^1\}\in X_1,\ \{x_m^2\}\in X_2,\ \{x_m^3\}\in X_3,\ldots,\{x_m^n\}\in X_n$. Now choose a point $x_1^2\in T_1(x_1^1)$ then a point $x_1^3\in T_1(x_1^2)$, then a point $x_1^4\in T_3(x_1^3),\ldots,x_2^1\in T_n(x_1^{n-1})$.

In general having chosen $x_m^1\in X_1,x_m^2\in X_2,\ldots,x_m^n\in X_n$. Choose a point $x_{m+1}^1\in T_n(x_m^n)$ then a point $x_{m+1}^2\in T_1(x_{m+1}^1)$ and then a point $x_{m+1}^3\in T_2(x_{m+1}^2)$ and continue like this $x_{m+1}^n\in T_{n-1}(x_{m+1}^{n-1})$. For $m=1,2,3,\ldots$

Applying in inequality (5), we get,

$$\begin{aligned}
d_1\left(x_{m+1}^1, x_{m+2}^1\right) &\leqslant \delta_1\left(T_n T_{n-1} \ldots T_2 T_1 x_m^1, T_n T_{n-1} \ldots T_2 T_1 x_{m+1}^1\right) \\
&\leqslant c \max \left\{ d_1\left(x_m^1, x_{m+1}^1\right),\right. \\
&\quad \delta_1\left(x_m^1, T_n \ldots T_2 T_1 x_m^1\right), \\
&\quad \delta_1\left(x_{m+1}^1, T_n \ldots T_2 T_1 x_{m+1}^1\right), \\
&\quad \delta_2\left(T_1 x_m^1, T_1 x_{m+1}^1\right), \\
&\quad \delta_3\left(T_2 T_1 x_m^1, T_2 T_1 x_{m+1}^1\right), \ldots, \\
&\quad \delta_n\left(T_{n-1} T_{n-2} \ldots T_2 T_1 x_m^1,\right. \\
&\quad \left.\left. T_{n-1} T_{n-2} \ldots T_2 T_1 T_n x_{m+1}^1\right)\right\} \\
&= c \max \left\{ d_1\left(x_m^1, x_{m+1}^1\right),\right. \\
&\quad \delta_1\left(x_m^1, x_{m+1}^1\right), \delta_1\left(x_{m+1}^1, T_n \ldots T_2 T_1 x_{m+1}^1\right), \\
&\quad \delta_2\left(T_1 T_n \ldots T_2 x_{m-1}^2, T_1 T_n \ldots T_2 x_m^2\right), \\
&\quad \delta_3\left(T_2 T_1 T_n \ldots T_3 x_{m-1}^3, T_2 T_1 T_n \ldots T_3 x_m^3\right), \ldots, \\
&\quad \delta_n\left(T_{n-1} \ldots T_2 T_1 x_{m-1}^n,\right. \\
&\quad \left.\left. T_{n-1} T_{n-2} \ldots T_2 T_1 T_n x_m^n)\right\}\right.
\end{aligned}$$

So we get,

$$\begin{aligned}
d_1\left(x_{m+1}^1, x_{m+2}^1\right) \leqslant c \max &\left\{ \delta_1\left(T_n T_{n-1} \ldots T_2 T_1 x_{m-1}^1,\right.\right. \\
&\quad \left. T_n \ldots T_2 T_1 x_m^1\right), \\
&\quad \delta_2\left(T_1 T_n \ldots T_2 x_{m-1}^2, T_1 T_n \ldots T_2 x_m^2\right), \\
&\quad \delta_3\left(T_2 T_1 T_n \ldots T_3 x_{m-1}^3, T_2 T_1 T_n \ldots T_3 x_m^3\right) \\
&\quad , \ldots, \delta_n\left(T_{n-1} \ldots T_2 T_1 x_{m-1}^n,\right. \\
&\quad \left.\left. T_{n-1} T_{n-2} \ldots T_2 T_1 T_n x_m^n)\right\}\right.
\end{aligned} \tag{9}$$

Applying inequality (6), we obtain,

$$\begin{aligned}
d_2\left(x_{m+1}^2, x_{m+2}^2\right) &\leqslant \delta_2\left(T_1T_n \ldots T_3T_2x_m^2, T_1T_n \ldots T_3T_2x_{m+1}^2\right) \\
&\leqslant c\max\left\{\delta_2\left(x_m^2, x_{m+1}^2\right),\right. \\
&\quad \delta_2\left(x_m^2, T_1T_n \ldots T_2x_m^2\right), \\
&\quad \delta_2\left(x_{m+1}^2, T_1T_n \ldots T_2x_{m+1}^2\right), \\
&\quad \delta_3\left(T_2x_m^2, T_2x_{m+1}^2\right), \\
&\quad \delta_4\left(T_3T_2x_m^2, T_3T_2x_{m+1}^2\right), \ldots, \\
&\quad \delta_n\left(T_{n-1}T_{n-2} \ldots T_2x_m^2,\right. \\
&\quad \left. T_{n-1}T_{n-2} \ldots T_2x_{m+1}^2\right) \\
&\quad \left.\delta_1\left(T_nT_{n-1} \ldots T_2x_m^2, T_nT_{n-1} \ldots T_2x_{m+1}^2\right)\right\} \\
&\leqslant c\max\left\{\delta_2\left(x_m^2, T_1T_n \ldots T_2x_m^2\right),\right. \\
&\quad \delta_3\left(T_2T_1T_n \ldots T_3x_{m-1}^3, T_2T_1T_n \ldots T_3x_m^3\right), \\
&\quad \delta_4\left(T_3T_2T_1T_n \ldots T_4x_{m-1}^4, T_3T_2T_1T_n \ldots T_4x_m^4\right), \\
&\quad \delta_n\left(T_{n-1}T_{n-2} \ldots T_2T_1T_nx_{m-1}^n,\right. \\
&\quad \left. T_{n-1}T_{n-2} \ldots T_2T_1T_nx_m^n\right), \\
&\quad \left.\delta_1\left(T_nT_{n-2} \ldots T_2T_1x_{m-1}^1, T_nT_{n-2} \ldots T_2T_1x_m^1\right)\right\}
\end{aligned}$$

So we get,

$$\begin{aligned}
d_2\left(x_{m+1}^2, x_{m+2}^2\right) \leqslant c\max&\left\{\delta_1\left(T_n \ldots T_2T_1x_{m-1}^1, T_n \ldots T_2T_1x_m^1\right),\right. \\
&\delta_2\left(T_1T_n \ldots T_2x_{m-1}^2, T_1T_n \ldots T_2\left(x_m^2\right)\right), \\
&\delta_3\left(T_2T_1T_n \ldots T_3x_{m-1}^3,\right. \\
&\left. T_2T_1T_nT_{n-1} \ldots T_3x_m^3\right), \\
&\ldots, \delta_n\left(T_{n-1}T_{n-2} \ldots T_2T_1T_nx_{m-1}^n,\right. \\
&\left.\left. T_{n-1}T_{n-2} \ldots T_2T_1T_nx_m^n\right)\right\}
\end{aligned} \tag{10}$$

Applying inequality (7), we get

$$
\begin{aligned}
d_3\left(x_{m+1}^3, x_{m+2}^3\right) \leqslant\, & \delta_3\left(T_2T_1T_n\ldots T_3x_m^3, T_2T_1T_n\ldots T_3x_{m+1}^3\right) \\
\leqslant\, & c\max\left\{d_3\left(x_m^3, x_{m+1}^3\right), \delta_3\left(x_m^3, x_{m+1}^3\right),\right. \\
& \delta_3\left(x_{m+1}^3, x_{m+2}^3\right), \\
& \delta_4\left(T_3T_2T_1T_n\ldots T_4x_{m-1}^4, T_3T_2T_1T_n\right), \\
& \delta_3\left(x_{m+1}^3, T_2T_1T_n\ldots T_4T_3x_{m+1}^3\right), \\
& \delta_5\left(T_4T_3T_2T_1T_n\ldots T_5x_{m-1}^5, T_4T_3T_2T_1\ldots T_5x_m^5\right), \\
& \ldots, \delta_n\left(T_{n-1}\ldots T_4T_3T_2T_1T_nx_{m-1}^n,\right. \\
& \left.T_{n-1}\ldots T_1T_nx_m^n\right), \\
& \delta_1\left(T_nT_{n-1}\ldots T_4T_3T_2T_1x_{m-1}^1, T_n\ldots T_2T_1x_m^1\right), \\
& \left.\delta_2\left(T_1T_n\ldots T_4T_3T_2x_{m-1}^2, T_1T_n\ldots T_4T_3T_2x_m^2\right)\right\}
\end{aligned}
$$

So we get,

$$
\begin{aligned}
d_3\left(x_{m+1}^3, x_{m+2}^3\right) \leqslant\, & c\max\left\{\delta_3\left(T_2T_1T_n\ldots T_3x_{m-1}^3,\right.\right. \\
& \left.T_2T_1T_n\ldots T_3x_m^3\right), \\
& \delta_1\left(T_nT_{n-1}\ldots T_2T_1x_{m-1}^1,\right. \\
& \left.T_nT_{n-1}\ldots T_2T_1x_m^1\right), \\
& \delta_2\left(T_1T_n\ldots T_2x_{m-1}^2, T_1T_n\ldots T_2x_m^2\right), \ldots \\
& \left.\delta_n\left(T_{n-1}\ldots T_1T_nx_{m-1}^n, T_{n-1}\ldots T_1T_nx_m^n\right)\right\} \\
& \vdots
\end{aligned} \tag{11}
$$

Continuously like this, using inequality (8), we get,

$$
\begin{aligned}
d_n\left(x_{m+1}^n, x_{m+n}^n\right) \leqslant\, & \delta_n\left(T_{n-1}T_{n-2}\ldots T_1T_nx_m^n,\right. \\
& \left.T_{n-1}T_{n-2}\ldots T_1T_nx_m^n\right) \\
\leqslant\, & c\max\left\{d_n\left(x_m^n, x_{m+1}^n\right), \delta_1\left(T_nT_{n-1}\ldots T_nx_m^n\right),\right. \\
& \delta_n\left(x_{m+1}^n, T_{n-1}\ldots T_n\left(x_{m+1}^n\right)\right), \\
& \delta_1\left(T_nT_{n-1}\ldots T_2T_1x_{m-1}^1, T_n\ldots T_1x_m^1\right), \\
& \delta_3\left(T_2T_1T_nT_{n-1}\ldots T_3x_{m-1}^3,\right. \\
& \left.T_2T_1T_nT_{n-1}\ldots T_3x_m^3\right), \ldots, \\
& \delta_{n-1}\left(T_{n-2}\ldots T_1T_nT_{n-1}x_{m-1}^{n-1},\right. \\
& \left.\left.T_{n-2}\ldots T_1T_nT_{n-1}x_m^{n-1}\right)\right\}
\end{aligned}
$$

So we get,

$$
\begin{aligned}
d_n\left(x^n_{m+1}, x^n_{m+2}\right) \leqslant c \max \{ & \delta_1\left(T_nT_{n-1}\dots T_2T_1x^1_{m-1},\right. \\
& \left. T_nT_{n-1}\dots T_2T_1x^1_m\right), \\
& \delta_1\left(T_1T_n\dots T_2x^2_{m-1}, T_1T_n\dots T_2x^2_m\right), \\
& \dots, \delta_n\left(T_{n-1}T_{n-2}\dots T_1T_nx^n_{m-1},\right. \\
& \left. T_{n-1}T_{n-2}\dots T_1T_nx^n_m)\right\}
\end{aligned} \tag{12}
$$

So we get by using above inequality. It now follow easily by induction on using inequalities (9), (10), (11) and (12).

$$
\begin{aligned}
d_1\left(x^1_{m+1}, x^1_{m+2}\right) \leqslant c^{m-1} \max \{ & \delta_1\left(T_nT_{n-1}\dots T_2T_1x^1_1,\right. \\
& \left. T_n\dots T_2T_1x^1_2\right), \\
& \delta_2\left(T_1T_n\dots T_2x^2_1, T_1T_n\dots T_2x^2_2\right), \\
& \delta_3\left(T_2T_1T_n\dots T_3x^3_1, T_2T_1T_n\dots T_3x^3_2\right), \\
& \dots, \delta_n(T_{n-1}T_{n-2}\dots T_2T_1T_nx^n_1, \\
& T_{n-1}T_{n-2}\dots T_2T_1T_nx^n_2)\} \\
d_2\left(x^2_{m+1}, x^2_{m+2}\right) \leqslant c^{m-1} \max \{ & \delta_1\left(T_nT_{n-1}\dots T_1x^1_1, T_n\dots T_1x^1_2\right), \\
& \delta_2\left(T_1T_n\dots T_2x^2_1, T_1T_n\dots T_2x^2_2\right), \dots, \\
& \delta_n(T_{n-1}T_{n-2}\dots T_2T_1T_nx^n_1, \\
& T_{n-1}\dots T_2T_1T_nx^n_2)\} \\
& \vdots \\
d_n\left(x^n_{m+1}, x^n_{m+2}\right) \leqslant c^{m-1} \max & \\
& \{\delta_1\left(T_nT_{n-1}\dots T_1x^1_1, T_n\dots T_1x^1_2\right), \\
& \delta_2\left(T_1T_n\dots T_2x^2_1, T_1T_n\dots T_2x^2_2\right), \dots, \\
& \delta_n(T_{n-1}T_{n-2}\dots T_1T_nx^n_1, \\
& T_{n-1}T_{n-2}\dots T_1T_nx^n_2)\}
\end{aligned} \tag{13}
$$

Then for $r = 1, 2, \ldots$ and arbitrary $\varepsilon > 0$ we have,

$$\begin{aligned} d_1\left(x^1_{m+1}, x^1_{m+r+1}\right) &\leqslant \delta_1\left(T_n T_m \ldots T_2 T_1 x^1_m, T_n \ldots T_1 x^1_{m+r}\right) \\ &\leqslant \delta_1\left(T_n T_{n-1} \ldots T_1 x^1_m, T_n T_{n-1} \ldots T_1 x^1_{m+1}\right) \\ &\quad + \delta_1\left(T_n T_{n-1} \ldots T_1 x^1_{m+1},\right. \\ &\quad \left.T_n T_{n-1} \ldots T_1 x^1_{m+2}\right) + \cdots + \\ &\quad \delta_1\left(T_n T_{n-1} \ldots T_1 x^1_{m+r-1},\right. \\ &\quad \left.T_n T_{n-1} \ldots T_1 x^1_{m+r}\right)\Big\} \\ &\leqslant \left(c^m + c^{m+1} + \cdots + c^{m+r-1}\right) \qquad (14) \\ &\quad \times \max\Big\{d_1\left(T_n \ldots T_1 x^1_1, T_n \ldots T_1 x^1_2\right), \\ &\quad \delta_2\left(T_1 T_n \ldots T_2 x^2_1, T_1 T_n \ldots T_2 x^2_2\right), \\ &\quad \delta_3\left(T_2 T_1 T_n \ldots T_3 x^3_1, T_2 T_1 T_n \ldots T_3 x^3_2\right), \\ &\quad \ldots, \\ &\quad \delta_n\left(T_{n-1} \ldots T_1 T_n x^n_1, T_{n-1} \ldots T_1 T_n x^n_2\right)\Big\} \\ &< \varepsilon \end{aligned}$$

For m greater than some N since $c < 1$. The sequence $\{x^1_m\}$ is therefore a Cauchy sequence in complete metric space X_1 and so it has limit $\alpha_1 \in X_1$, In the parallel manner, $\{x^2_m\}$ has limit $\alpha_2 \in X_2, \ldots, \{x^2_m\}$ has limit $\alpha_n \in X_n$.

Further inequality (14) gives,

$$\begin{aligned} \delta_1\left(\alpha_1, T_n T_{n-1} \ldots T_2 T_1 x^1_m\right) &\leqslant d_1\left(\alpha_1, x^1_{l+1}\right) + \\ &\quad \delta_1\left(x^1_{l+1}, T_n T_{n-1} \ldots T_2 T_1 x^1_m\right) \\ &\leqslant d_1\left(\alpha_1, x^1_{l+1}\right) + \delta_1\left(T_n T_{n-1} \ldots T_2 T_1 x^1_l,\right. \\ &\quad \left.T_n T_{n-1} \ldots T_2 T_1 x^1_m\right), \\ &\leqslant d_1\left(\alpha_1, x^1_{l+1}\right) + \varepsilon \end{aligned}$$

For $m, l \geqslant N$, letting $l \to \infty$ it follows that

$$\delta_1\left(\alpha_1, T_n T_{n-1} \ldots T_2 T_1 x^1_m\right) \leqslant \varepsilon$$

For $m > N$ and so we get,

$$\begin{aligned} \lim_{m\to\infty} T_n T_{n-1} \ldots T_2 T_1 x^1_m &= \{\alpha_1\} \\ &= \lim_{m\to\infty} T_n T_{n-1} \ldots T_2\left(x^2_m\right) \qquad (15) \\ \Longrightarrow \qquad T_n\{\alpha_n\} &= \{\alpha_1\} \end{aligned}$$

Since ε is arbitrary.

$$
\begin{aligned}
\lim_{m\to\infty} T_1 T_n \ldots T_3 T_2 \left(x_m^2\right) &= \{\alpha_2\} \\
&= \lim_{m\to\infty} T_1 T_n \ldots T_3 x_m^3 \\
\lim_{m\to\infty} T_2 T_1 T_n \ldots T_4 T_3 x_m^3 &= \{\alpha_3\} \\
&= \lim_{m\to\infty} T_2 T_1 T_n \ldots T_4 x_m^4 \\
&\;\;\vdots \\
\lim_{m\to\infty} T_{n-2} T_{n-3} \cdots & \\
T_1 T_n T_{n-1} \left(x_m^{n-1}\right) &= \{\alpha_{n-1}\} \\
&= \lim_{m\to\infty} T_{n-1} T_{n-2} \ldots T_1 \left(x_{m+2}^1\right)
\end{aligned}
\tag{16}
$$

Now using the continuity of $T_1, T_2, T_3 \ldots, T_{n-1}$ we obtain,

$$
\begin{aligned}
\lim_{m\to\infty} x_m^2 &= \lim_{m\to\infty} T_1 x_m^1 = T_1 \alpha_1 = \alpha_2 \\
\lim_{m\to\infty} x_m^3 &= \lim_{m\to\infty} T_2 x_m^2 = T_2 \alpha_2 = \alpha_3 \\
&\;\;\vdots \\
\lim_{m\to\infty} x_m^n &= \lim_{m\to\infty} T_{n-1} \left(x_m^{n-1}\right) \\
&= T_{n-1} \left(\alpha_{n-1}\right) = \alpha_n
\end{aligned}
$$

and then we see that

$$
T_{n-1} \left(T_{n-2} T_{n-3} \ldots T_2 T_1\right) \left(\alpha_1\right) = \alpha_n \tag{17}
$$

Now show that α_1 fixed point of $T_n T_{n-1} \ldots T_2 T_1$

Now applying inequality (5), we have,

$$
\begin{aligned}
d_1 \left(T_n T_{n-1} \ldots T_2 T_1 \alpha_1, x_m^1\right) &\leqslant \delta_1 \left(T_n T_{n-1} \ldots T_2 T_1 \alpha_1, T_n \ldots T_2 T_1 x_m^1\right) \\
&\leqslant c \max \big\{ d_1 \left(\alpha_1, x_{m-1}^1\right), \\
&\quad \delta_1 \left(\alpha_1, T_n T_{n-1} \ldots T_2 T_1 \alpha_1\right), \\
&\quad \delta_1 \left(x_{m-1}^1, T_n T_{n-1} \ldots T_2 T_1 x_{m-1}^1\right), \\
&\quad \delta_2 \left(T_1 \left(\alpha_1\right), T_1 \left(x_{m-1}^1\right)\right), \\
&\quad \delta_3 \left(T_2 T_1 \alpha_1, T_2 T_1 x_{m-1}^1\right), \ldots, \\
&\quad \delta_n \left(T_{n-1} T_{n-2} \ldots T_2 T_1 \alpha_1,\right. \\
&\quad \left. T_{n-1} \ldots T_2 T_1 x_{m-1}^1\right) \big\}
\end{aligned}
$$

Since $T_1, T_2, \ldots, T_{n-1}$ are continuous and $m \to \infty$ and using (16),

$$\delta_1 \left(T_n T_{n-1} \ldots T_2 T_1 \alpha_1, \alpha_1\right) \leqslant c\delta_1 \left(\alpha_1 T_n T_{n-1} \ldots T_2 T_1 \left(\alpha_1\right)\right)$$

Since $0 \leqslant c < 1$ so we have,

$$T_n T_{n-1} \ldots T_2 T_1 \left(\alpha_1\right) = \alpha_1$$

So α_1 is a fixed point of $T_n T_{n-1} \ldots T_2 T_1$. So we have,

$$T_n T_{n-1} \ldots T_2 T_1 \left(\alpha_1\right) = T_n \left(\alpha_n\right) = \alpha_1$$

On using equation,

$$\begin{aligned} T_1 T_n \ldots T_3 T_2 \left(\alpha_2\right) &= T_1 T_n \ldots T_3 T_2 T_1 \alpha_1 \\ &= T_1 \alpha_1 \\ &= \alpha_2 \end{aligned}$$

So
$$T_1 T_n \ldots T_3 T_2 \alpha_2 = \alpha_2$$

Similarly,

$$\begin{aligned} T_2 T_1 T_n \ldots T_4 T_3 \left(\alpha_3\right) &= \alpha_3 \\ T_3 T_2 T_1 T_n \ldots T_5 T_4 \left(\alpha_4\right) &= \alpha_4 \\ &\vdots \\ T_{n-1} T_{n-2} \ldots T_1 T_n \left(\alpha_n\right) &= T_{n-1} T_{n-2} \ldots T_1 T_n T_{n-1} \alpha_{n-1} \\ &= T_{n-1} \left(\alpha_{n-1}\right) \\ &= \alpha_n \end{aligned}$$

So we get α_1 is fixed point of $T_n T_{n-1} \ldots T_2 T_1$, and $\alpha_1 \in X_1$, α_2 is fixed point of $T_1 T_n \ldots T_3 T_2$ and $\alpha_2 \in X_2$ and continuous like the we get α_n s fixed point of $T_{n-1} T_{n-2} \ldots T_1 T_n$.

Uniqueness

To prove the uniqueness of α_1, let us suppose that
$T_n T_{n-1} \ldots T_2 T_1$ has second fixed point α_1' such that $\alpha_1 \neq \alpha_1'$ and $T_n T_{n-1} \ldots T_2 T_1 \alpha_1' =$

α'_1. Now using inequality (5), we have,

$$\begin{aligned} d_1\left(\alpha_1, \alpha'_1\right) &= \delta_1\left(T_nT_{n-1}\dots T_2T_1\alpha_1, T_nT_{n-1}\dots T_2T_1\alpha'_1\right) \\ &\leqslant c\max\left\{d_1\left(\alpha_1, \alpha'_1\right), \delta_1\left(\alpha_1, T_nT_{n-1}\dots T_2T_1\alpha_1\right),\right. \\ &\quad \delta_1\left(\alpha'_1, T_nT_{n-1}\dots T_2T_1\alpha'_1\right), \delta_2\left(T_1\alpha_1, T_1\alpha'_1\right), \\ &\quad \delta_3\left(T_2T_1\alpha_1, T_2T_1\alpha'_1\right), \dots, \\ &\quad \left.\delta_n\left(T_{n-1}\dots T_2T_1\alpha_1, T_{n-1}\dots T_2\alpha'_1\right)\right\} \\ &= c\max\left\{d_2\left(T_1\alpha_1, T_1\alpha'_1\right), d_3\left(T_2T_1\alpha_1, T_2T_1\alpha'_1\right),\right. \\ &\quad \left.\dots, d_n\left(T_{n-1}\dots T_2T_1\alpha_1, T_{n-1}\dots T_2T_1\alpha'_1\right)\right\} \end{aligned}$$

Next using inequality (6), we have

$$\begin{aligned} d_2\left(T_1\alpha_1, T_1\alpha'_1\right) &= \delta_2\left(T_1T_n\dots T_3T_2T_1\alpha_1, T_1T_n\dots T_3T_2T_1\alpha'_1\right) \\ &\leqslant c\max\left\{d_2\left(T_1\alpha_1, T_1\alpha'_1\right),\right. \\ &\quad d_2\left(T_1\alpha_1, T_1T_n\dots T_3T_2T_1\alpha_1\right), \\ &\quad d_2\left(T_1\alpha'_1, T_1T_n\dots T_3T_2T_1\alpha'_1\right), \\ &\quad d_3\left(T_2T_1\alpha_1, T_2T_1\alpha'_1\right), \\ &\quad d_4\left(T_3T_2T_1\alpha'_1, T_3T_2T_1\alpha_1\right), \dots, \\ &\quad d_n\left(T_{n-1}T_{n-2}\dots T_3T_2T_1\alpha_1,\right. \\ &\quad \left.T_{n-1}T_{n-2}\dots T_3T_2T_1\alpha'_1\right), \\ &\quad \left.d_1\left(T_nT_{n-1}\dots T_2T_1\alpha_1, T_nT_{n-1}\dots T_2T_1\alpha'_1\right)\right\} \\ &= c\max\left\{d_3\left(T_2T_1\alpha_1, T_2T_1\alpha'_1\right),\right. \\ &\quad d_4\left(T_3T_2T_1\alpha_1, T_3T_2T_1\alpha'_1\right), \dots, \\ &\quad d_n\left(T_{n-1}T_{n-2}\dots T_3T_2T_1\alpha_1,\right. \\ &\quad \left.\left.T_{n-1}T_{n-2}\dots T_3T_2T_1\alpha'_1\right), d_1\left(\alpha_1, \alpha'_1\right)\right\} \end{aligned}$$

Similarly by using (7), we get

$$\begin{aligned} d_3\left(T_2T_1\alpha_1, T_2T_1\alpha'_1\right) \leqslant c\max&\left\{d_4\left(T_3T_2T_1\alpha_1, T_3T_2T_1\alpha'_1\right),\right. \\ &d_5\left(T_4T_3T_2T_1\alpha_1, T_4T_3T_2T_1\alpha'_1\right), \dots, \\ &d_n\left(T_{n-1}T_{n-2}\dots T_3T_2T_1\alpha_1,\right. \\ &\left.T_{n-1}T_{n-2}\dots T_3T_2T_1\alpha'_1\right), \\ &\left.d_1\left(\alpha_1, \alpha'_1\right), d_2\left(T_1\alpha_1, T_1\alpha'_1\right)\right\} \end{aligned}$$

Now using (8), we get

$$d_n\left(T_{n-1}T_{n-2}\dots T_2T_1\alpha_1,\ T_{n-1}T_{n-2}\dots T_2T_1\alpha'_1\right)$$
$$\leqslant c\max\left\{d_1\left(\alpha_1,\alpha'_1\right), d_2\left(T_1\alpha_1,T_1\alpha'_1\right),\ d_3\left(T_2T_1\alpha_1,T_2T_1\alpha'_1\right),\dots,\ d_{n-1}\left(T_{n-2}T_{n-3}\dots T_2T_1\alpha_1,\ T_{n-2}T_{n-3}\dots T_2T_1\alpha'_1\right)\right\}$$

It now follows from inequalities that, $d_1\left(\alpha,\alpha'_1\right)=0$ proving that uniqueness of α_1. The uniqueness of $\alpha_2,\alpha_3,\dots,\alpha_n$ follows similarly.

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