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LIE ALGEBRA BUNDLES OF FINITE TYPE

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Abstract. In this paper, we study some homotopic properties of Lie algebra bundles over a compact Hausdorff space. Further we give a bijection between Lie algebra bundles of finite type over a general topological space and finitely generated projective Lie rings over the ring of continuous functions on the base space.

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1. Introduction

A weak Lie algebra bundle is a vector bundle $\xi = (\xi, p, X)$, together with a morphism $\theta : \xi \oplus \xi \to \xi$ which induces Lie algebra structure on each fibre ξ_x.

A Lie algebra bundle is a vector bundle $\xi = (\xi, p, X)$ in which each fibre is Lie algebra and for each x in X there exist an open neighborhood U of x in X, a Lie algebra L and a homeomorphism $\Phi : U \times L \to p^{-1}(U)$ such that restriction $\Phi_x : x \times L \to p^{-1}(x)$ is Lie algebra isomorphism.

A subalgebra bundle of a Lie algebra bundle is a vector subbundle in which each fibre is a subalgebra. Further if each fibre is an ideal then it is called an ideal bundle.

A morphism $\varphi : \xi \to \zeta$ of Lie algebra bundles ξ and ζ over the same base space X is continuous and for each x in X, $\varphi_x : \xi_x \to \zeta_x$ is a Lie algebra homomorphism. A morphism φ is an isomorphism if φ is bijective and φ^{-1} is continuous.

Conventions: All the underlying vector spaces of the fibres are finite dimensional over a field of characteristic zero. All our bundles and subbundles are over the compact Hausdorff space unless otherwise stated.

2. Homotopic properties of Lie algebra bundles

In [6], we defined the pullback of a Lie algebra bundle to be

Definition 1. *Let ξ be a Lie algebra bundle over X and $f : Y \to X$ is a continuous map, then we define pullback of ξ to be $f^*(\xi) = (\xi^*, p^*, Y)$, where $\xi^* = \{(y, u) \in Y \times \xi | f(y) = p(u)\}$.*

We also have obtained the following results.

Lemma 1. *Let ξ be a Lie algebra bundle over a topological space X, and let $g : Y \to X$ be a continuous map. Then, the induced bundle $g^*(\xi)$ is a Lie algebra bundle over Y.*

Lemma 2. *Let Y be closed subspace of a compact space X, and ξ is a Lie algebra bundle over X. Then any section $s : Y \to \xi|_Y$ can be extended to X.*

Lemma 3. *Let Y be a closed subspace of a compact space X, and let ξ and η are Lie algebra bundles over X. If $f : \xi|_Y \to \eta|_Y$ is an isomorphism, then there exists an open set U containing Y and an extension $f : \xi|_U \to \eta|_U$ which is an isomorphism.*

Lemma 4. *Let Y be a compact Hausdorff space, $f_t : Y \to X$ $(0 \le t \le 1)$ a homotopy and let ξ be a Lie algebra bundle over X. Then $f_0^*(\xi) \cong f_1^*(\xi)$.*

Now we proceed to obtain few more results in this direction.

Lemma 5. *Given a pair of continuous maps $Z \xrightarrow{f} Y \xrightarrow{g} X$ and a Lie algebra bundle $\xi = (\xi, p, X)$ there is an isomorphism of Lie algebra bundles between $f^*(g^*(\xi))$ and $(g \circ f)^*(\xi)$ over Z.*

Proof. We have

$$(g \circ f)^*(\xi) = \{(z, u) \in Z \times \xi |\ g(f(z)) = p(u)\}$$

and

$$\begin{aligned} f^*(g^*(\xi)) &= \{(z, v) \in Z \times g^*(\xi) \mid f(z) = q(v)\} \\ &= \{(z, (y, v)) \in Z \times (Y \times \xi) \mid g(y) = p(v), f(z) = y\} \\ &= \{z, (f(x), v) \mid g(f(z)) = p(v)\}. \end{aligned}$$

The map $(z, u) \to (z, (f(z), u))$ is a Lie algebra bundle isomorphism from $(g \circ f)^*(\xi)$ to $f^*(g^*(\xi))$.

Let $\mathrm{Lie}(X)$ denote the isomorphism class of Lie algebra bundles over X.

Lemma 6. *If $f : X \to Y$ is a homotopy equivalence, then $f^* : Lie(Y) \to Lie(X)$ is bijective. If X is contractible, every Lie algebra bundle over X is trivial bundle.*

Proof. Consider a Lie algebra bundle ξ over X and η a Lie algebra bundle over Y.

Let g be the homotopic inverse of f, then there exist a homotopy $H : X \times I \to X$ such that $H|_{X\times\{0\}} = g \circ f$ and $H|_{X\times\{1\}} = Id_X$. Then by Lemma (4) we have $H_0^*(\xi) \cong H_1^*(\xi)$, i.e., $(g \circ f)^*(\xi) \cong \xi$, and so $f^* \circ g^*(\xi) \cong \xi$. Similary we can have a homotopy $G : Y \times I \to Y$ such that $G|_{Y\times\{0\}} = f \circ g$ and $G|_{Y\times\{1\}} = Id_Y$, with $G_0^*(\eta) = G_1^*(\eta)$, i.e., $(f \circ g)^*(\eta) \cong \eta$, and so $g^* \circ f^*(\eta) \cong \eta$. Thus f^* has two sided inverse and hence f^* is bijective.

For the second part we have the space X is contractible, so we have a homotopy $H : X \times I \to X$ such that $H|_{X\times\{0\}} = Id_X$ and $H|_{X\times\{1\}} = x_0$ (a constant map). Since $H_0^*(\xi) = H_1^*(\xi)$, but $H_0^*(\xi) = \xi$ and

$$\begin{aligned} H_1^*(\xi) &= \bigcup_{x\in X} \xi_{x_0} \\ &= X \times \xi_{x_0} \end{aligned}$$

Thus any Lie algebra bundle over a contractible space X is a trivial bundle.

Lemma 7. *If ξ is a Lie algebra bundle over $X \times I$, and $\pi : X \times I \to X \times \{0\}$ is the projection, ξ is isomorphic to $\pi^*(\xi|_{X\times 0})$.*

Proof. Define a homotopy $H : (X \times I) \times I \to X \times I$ by $H(x, s, t) = (x, st)$. Then $H|_{X\times I\times\{0\}} = \pi$ and $H|_{X\times I\times\{1\}} = Id_{X\times I}$. Thus

$$\pi^*(\xi|_{X\times 0}) = H_0^*(\xi) \cong H_1^*(\xi) = \xi$$

Definition 2. *If Y is a closed subspace of a compact space X and ξ is a Lie algebra bundle over X and $\alpha : \xi|_Y \to Y \times L$ is an isomorphism, then we call α is a trivialization of ξ over Y.*

Lemma 8. *A trivialization α of a Lie algebra bundle ξ over $Y \subseteq X$ defines a Lie algebra bundle ξ/α over X/Y. The isomorphism class of ξ/α depends only on the homotopy class α.*

Proof. Step 1: To define the Lie algebra bundle ξ/α over X/Y, let us consider the projection $\pi : Y \times L \to L$ and we define an equivalence relation on $\xi|_Y$ as follows, for any $e, e' \in \xi/Y$ by $e \sim e'$ if and only if $\pi\alpha(e) = \pi\alpha(e')$. We extend this relation to $\xi/X - Y$ by $e \sim e'$ if and only if $e = e'$. Let ξ/α denote the set of all equivalence classes induced by this equivalence relation with the quotient topology.

Since X/Y is obtained by considering Y as a single point $\{Y\}$. We need to verify the local triviality at the base point $\{Y\}$ of X/Y.

As α is a trivialization of ξ over Y, $\alpha : \xi|_Y \to Y \times L$ is an isomorphism. Since X is a compact Hausdorff and Y is a closed subspace of X, there exist an open set U containing Y and an isomorphism $\alpha : \xi|_U \to U \times L$. Then it will induce an isomorphism

$$(\xi/\alpha)|_{U|_Y} \cong (U|_Y) \times L,$$

which will give the local triviality at Y. Thus ξ/α is an Lie algebra bundle over X/Y.

Step 2: Suppose α_0 and α_1 are homotopic trivializations of ξ over Y. Then there exist a homotopy trivialization β of $\xi \times I$ over $Y \times I \subseteq X \times I$ such that

$$\alpha_0 = \beta|_{Y\times\{0\}} : \xi \times I|_{Y\times\{0\}} \to Y \times \{0\} \times L$$

and

$$\alpha_1 = \beta|_{Y\times\{1\}} : \xi \times I|_{Y\times\{1\}} \to Y \times \{1\} \times L.$$

Now consider $f : (X/Y) \times I \to (X \times I)/(Y \times I)$ be canonical map. Since β is a trivialization of $\xi \times I$ over $Y \times I$, $\xi \times I/\beta$ is a Lie algebra bundle over $(X \times I)/(Y \times I)$. Thus $f^*(\xi \times I)/\beta$ is a Lie algebra bundle over $(X/Y) \times I$ whose restriction to $(X/Y) \times \{0\}$ is ξ/α_0 and $(X/Y) \times \{1\}$ is ξ/α_1. Since α_0 and α_1 are homotopic trivialization it follows that

$$\xi/\alpha_0 \cong \xi/\alpha_1$$

.

Consider ξ_i be Lie algebra bundle over X_i for $i = 1, 2$. Let $X = X_1 \bigcup X_2$ and $A = X_1 \bigcap X_2$. If $\varphi : \xi_1|_A \to \xi_2|_A$ is an isomorphism then we can construct the Lie algebra bundle $\xi = \xi_1 \bigcup_\varphi \xi_2$ over X.

We identify the element $e_1 \in \xi_1|_A$ with $\varphi(e_1) \in \xi_2|_A$. The topological space $\xi_1 \bigcup_\varphi \xi_2$ is the quotient of the disjoint union of ξ_1 and ξ_2 with the equivalence relation which identifies $e_1 \in \xi_1|_A$ with $\varphi(e_1) \in \xi_2|_A$. Then

$$\xi_1 \bigcup_{\varphi} \xi_2 = (\bigcup_{x \in X_1 - A} \xi_{1x}) \cup (\bigcup_{x \in X_2 - A} \xi_{2x}) \cup (\bigcup_{x \in A} \hat{\xi_x}),$$

where $\hat{\xi_x} = \{e \sim \varphi(e) |\ e \in \xi_{1x}\}$.

To check the local triviality of $\xi_1 \bigcup_\varphi \xi_2$:

The local triviality for points in $X_1 - A$ and $X_2 - A$ follows from that of ξ_1 and ξ_2 respectively. So we need to check the local triviality at the points of A.

Let $a \in A$, and V_1 be closed neighbourhood of a in X_1 over which ξ_1 is trivial. Then we have an isomorphism

$$\psi_1 : \xi_1|_{V_1} \to V_1 \times L.$$

Restricting this isomorphism to A, we have

$$\psi_1^A : \xi_1|_{V_1 \cap A} \to (V_1 \cap A) \times L.$$

Let $\psi_2^A = (\psi_1^A \circ \varphi^{-1})|_{V_1 \cap A}$. Then

$$\psi_2^A : \xi_2|_{V_1 \cap A} \to (V_1 \cap A) \times L$$

is an isomorphism. Since $V_1 \cap A$ is closed in X_2 there exists an open set V_2 containing $V_1 \cap A$ such that ψ_2^A extends to an isomorphism

$$\psi_2 : \xi_2|_{V_2} \to V_2 \times L.$$

Then ψ_1 and ψ_2 together with defines an isomorphism

$$\psi = \psi_1 \bigcup_{\varphi} \psi_2 : \xi_1 \bigcup_{\varphi} \xi_2|_{V_1 \cup V_2} \to V_1 \cup V_2 \times L,$$

given by

$$\psi(e) = \begin{cases} \psi_1(e), & \text{if } e \in \xi_{1x} \ x \in V_1 - A \\ \psi_2(e), & \text{if } e \in \xi_{2x} \ x \in V_2 - A \end{cases}$$

$$\psi(e, \varphi(e)) = \{ \ \psi_1(e) = \psi_2(\varphi(e)), \quad \text{if } e \in \xi_{1x} \ x \in A$$

which gives the local triviality of $\xi_1 \bigcup_\varphi \xi_2$

Lemma 9. *The isomorphism class of* $\xi_1 \bigcup_\varphi \xi_2$ *depends only on the homotopy class of the isomorphism* $\varphi : \xi_1|_A \to \xi_2|_A$.

Proof. A homotopy of isomorphisms $\varphi : \xi_1|_A \to \xi_2|_A$ is defined as an isomorphism

$$\Psi : \pi^*\xi_1|_{A \times I} \to \pi^*\xi_2|_{A \times I}$$

where $I = [0, 1]$ and $\pi : X \times I \to X$ is the projection map. Define a homotopy $f_t : X \to X \times I$ by $f_t(x) = (x, t)$ for all t in I. Let denote

$$\varphi_t : \xi_1|_A \to \xi_2|_A$$

the isomorphism induced from Ψ by f_t. Then

$$\xi_1 \bigcup_{\varphi_t} \xi_2 \cong f_t^*(\pi^*\xi_1 \bigcup_{\varphi} \pi^*\xi_2).$$

Since f_t's are homotopic we obtain

$$\xi_1 \bigcup_{\varphi_0} \xi_2 \cong \xi_1 \bigcup_{\varphi_1} \xi_2.$$

3. Lie algebra bundles and Projective modules

J.P. Serre [7] has shown that there is a one to one correspondence between algebraic vector bundles over an affine variety and finitely generated projective modules over its co-ordinate ring. Richard G. Swan [8] has shown that a similar correspondence exists between topological vector bundles over a compact Hausdorff space X and finitely generated projective modules over the ring of continuous real valued functions on X. Later Goodearl [3] observed that the equivalence holds in the more general case of paracompact Hausdorff space X if one restricts to the bundle of finite type (i.e. there exists a finite open covering T of X such that the restriction of the bundle to each $U \in T$ is trivial).

In 1986, L.N. Vaserstein [9] has extended this result with an appropriate definition of vector bundles of finite type to an arbitrary topological space X.

Definition 3. *A vector bundle over an arbitrary space X is of finite type if there is a finite partition S of 1 on X (that is a finite set S of nonnegative continuous functions on X whose sum is 1) such that the restriction of bundle to the set $\{x \in X | f(x) \neq 0\}$ is trivial for each f in S.*

Example 1. *Any vector bundle over a compact Hausdorff space X is of finite type .*

Now we proceed to give a bijection between Lie algebra bundles of finite type over a general topological space X and finitely generated projective Lie rings over the ring of continuous functions on X.

Definition 4. *A module P over a ring R is said to be a Lie ring if there exists a bilinear mapping $[\ ,\ ] : P \times P \to P$ such that*

1. $[u, u] = 0$ *for all* $u \in P$, *with* $Ch(R) \neq 2$,
2. $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$ *for all* u, v, w *in* P.

Remark 1. *Let us observe that the set of all sections of ξ denoted by $\Gamma(\xi)$ is a Lie ring, where $\xi = (\xi, p, X, \theta)$ is a Lie algebra bundle.*

Given two sections $S_1, S_2 : X \to \xi$ we define $[S_1, S_2] : X \to \xi$ by $[S_1, S_2](s) = \theta_s(S_1(s), S_2(s))$ for all $s \in X$. The mapping $[S_1, S_2] : X \to \xi$ is continuous since it is the following composition $X \xrightarrow{S_1 \oplus S_2} \xi \oplus \xi \xrightarrow{\theta} \xi$. Since θ is the Lie algebra structure on each fibre will satisfies the condition (i), (ii). Therefore $\Gamma(\xi)$ forms a Lie ring over $\mathcal{C}(X)$.

Theorem 10. *For every Lie ring P which is also finitely generated projective $\mathcal{C}(X)-$module, there is a Lie algebra bundle of finite type over X with $\Gamma(\xi) \cong P$.*

Proof. Since P is a finitely generated projective module over $\mathcal{C}(X)$, P is isomorphic to the column space of a square hermitian idempotent matrix $e = e^2 = e^*$ over $\mathcal{C}(X)$ [9]. Every finitely generated projective module P over $\mathcal{C}(X)$ gives an vector bundle ξ over X whose fibre at x is just $e(x)F^N$.

In [9] L.N.Vaserstein gave a general version of the proof which we will present here.

$\xi = \bigcup_{x\in X} e(x)F^N$ is locally trivial.

Let $x,\ y \in X$ and call $g(x,y) = e(x)e(y) + (I - e(x))(I - e(y))$, where I denotes the identity. Since $e(x),\ e(y)$ are idempotent, we can have

$$e(x)g(x,y) = e(x)e(y) = g(x,y)e(y).$$

Now fix x, since $g(x,x) = I$, there is an open neighbourhood U of x such that $g(x,y)$ is invertible for every $y \in U$.

For any $y \in U$ we have $e(y) = g(x,y)^{-1}e(x)g(x,y)$. So over U we get vector bundle isomorphism

$$U \times e(x)F^N \xrightarrow{\Phi} \xi|_U; \qquad (y,v) \xrightarrow{\Phi_y} (y, g(x,y)^{-1}v)$$

We now show that $\xi = \bigcup_{x\in X} e(x)F^N$ is vector bundle of finite type.

Let consider the set $Y = \{p \in M_n(F)|\ p = p^2 = p^*\}$ which is compact. Therefore there is a finite partition of unity $f_1, f_2, \cdots, f_n$ such that $|p - q| < \frac{1}{3}$ whenever $f_i(p)f_i(q) \neq 0$ for some i.

The matrix e above can be considered as a continuous map $X \to Y$, the $f_i \circ e$ make up a finite partition of unity on X.

Now we will show that ξ is trivial on every $U_i = \{x \in X | f_i \circ e(x) \neq 0\}$.

For any $x,\ y \in X$ we consider as above $g(x,y) = e(x)e(y) + (I - e(x))(I - e(y))$ with $e(x)g(x,y) = g(x,y)e(y)$. Also

$$\begin{aligned} g(x,y) &= e(x)e(y) + (I - e(x))(I - e(y)) \\ &= 2e(x)e(y) + I - e(x) - e(y) \\ &= I + (e(x) - e(y))(I - 2e(y)). \end{aligned}$$

Now, if $x, y \in U_i$ by definition of f_i and U_i we have

$$|g(x,y) - I| \leq |e(x) - e(y)||I - 2e(y)| < \frac{1}{3}.3 = 1$$

Thus $g(x,y)$ is invertible for $x, y \in U_i$.

Also for any $x, y \in U_i$, we have $e(y) = g(x,y)^{-1}e(x)g(x,y)$. As above we have the triviality on U_i.

We can give a Lie algebra bundle structure on ξ in the following way:

Let $I_x = \{\alpha \in \mathcal{C}(X) | \alpha(x) = 0\}$ be the maximal ideal of $\mathcal{C}(X)$ attached to $x \in X$. Then P/I_xP is isomorphic to $e(x)(F^N)$ given by the mapping $G_x : P/I_xP \to e(x)F^N$ defined by $G_x[e(f_1, f_2, \cdots, f_N) + I_xP] = e(x)(f_1(x), f_2(x), \cdots, f_N(x))$ which is an isomorphism of vector spaces [8].

Given two elements $e(x)(s),\ e(x)(t) \in e(x)F^N$ we can define the multiplication

$$\theta_x(e(x)(s), e(x)(t)) = G_x(G_x^{-1}(e(x)(s)) * G_x^{-1}(e(x)(t))),$$

where " $*$ " is the Lie multiplication on P. Hence $e(x)(F^N)$ has the structure of a Lie algebra as it inherits the Lie multiplication which we denote by θ_x from P and is having a vector space structure over F. Now let us define $\theta : \xi \to \xi$ as $\theta(u, v) = \theta_x(u, v)$, if u, v belong to $e(x)(F^N)$.

The continuity of θ follows from the commutative diagram

$$\begin{array}{ccc} U \times (e(x)F^N \times e(x)F^N) & \xrightarrow{\Phi\times\Phi} & \bigcup_{y\in U}(e(y)F^N \times e(y)F^N) \\ {\scriptstyle (I,\theta_x)}\big\downarrow & & \big\downarrow{\scriptstyle \theta} \\ U \times e(x)F^N & \xrightarrow{\Phi} & \bigcup_{y\in U} e(y)F^N \end{array}$$

Hence the theorem.

Now we proceed to prove the converse of the above Theorem.

Theorem 11. *If ξ is Lie algebra bundle of finite type over the base space X, then $\Gamma(\xi)$ is a Lie ring and finitely generated projective $\mathcal{C}(X)-$module.*

Proof. Suppose that P is of the form $\Gamma(\xi)$, where ξ is a Lie algebra bundle of finite type over X, then by P is a Lie ring by Remark (1) and also finitely generated projective $\mathcal{C}(X)-$module [9].

Theorem 12. *The functor Γ from the category of Lie algebra bundles over X and the category of finitely generated projective $\mathcal{C}(X)-$modules which are also Lie rings is an equivalence.*

Proof. From Theorems (10) and (11), Γ induces a bijective map of the isomorphism classes of the objects in these categories.Further, if $\varphi : \xi_1 \to \xi_2$ is a Lie algebra bundle isomorphism, then we define $\Gamma(\varphi) : \Gamma(\xi_1) \to \Gamma(\xi_2)$ by $\Gamma(\varphi)(S) = \varphi \circ S$, which is a Lie ring isomorphism. Finally given $\omega : P_1 \to P_2$ Lie ring isomorphism, there exist square hermitian idempotent matrices $e_1 = e_1^2 = e_1^*$ and $e_2 = e_2^2 = e_2^*$ over $\mathcal{C}(X)$ such that P_1 and P_2 are isomorphic to the column space of e_1 and e_2 respectively.

But corresponding Lie algebra bundle of finite type are $\xi_1 = \bigcup_{x \in X} e_1(x)F^N$ and $\xi_2 = \bigcup_{x \in X} e_2(x)F^N$. Define $\varphi : \xi_1 \to \xi_2$ by $\varphi|_{e_1(x)F^N} = e_2(x)F^N$.

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References

[1] M. F. Atiyah, *K-Theory*, W.A.Benjamin, Inc., New York, Amsterdam 1967.

[2] Douady, A. and Lazard, M., *Espace fibrés algèbres de Lie et en groupes*, Invent. Math., 1 (1966) 133-151.

[3] K. R. Goodearl, *Cancellation of Low-rank Vector Bundles*, Pasific J. Math., 113 (1984), 289-302.

[4] D. Husemoller, *Fibre Bundles*, McGraw Hill, New York 1966.

[5] B. S. Kiranagi, *Rings, Modules and Algebra Bundles,* Trends in Theory of Rings and Modules, Anamaya Publishers, New Delhi, India.

[6] R. Rajendra, B. S. Kiranagi and Ranjitha Kumar, *On Pullback Lie Algebra Bundles*, Adv.Studies Contemp. Math., 22, 4 (2012), 521-524.

[7] J. P. Serre, *Faiseaux Algebriques Coherent*, Ann. Of Math., 6, 2 (1955), 197-278.

[8] R. G. Swan, *Vector Bundles and Projective Modules*, Trans. Am. Math. Soc., 105 (1962), 264 - 277.

[9] L. N. Vaserstein, *Vector Bundles and Projective Modules*, Trans. Amer. Math., 294, 2 (1986), 749-755.

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