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FIXED COEFFICIENTS FOR A NEW SUBCLASS OF UNIFORMLY SPIRALLIKE FUNCTIONS

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Abstract. The main objective of this paper is to give several properties of the new subclass with negative coefficients and with fixed second coefficients.

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1. Introduction and Definitions

Let S denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic and univalent in the open unit disc $U = \{z \in \mathbb{C} : |z| \leq 1\}$. Also let S^* and $\mathcal{C}$ denote the subclasses of S that are respectively, starlike and convex. Motivated by certain geometric conditions, Goodman [1, 2] introduced an interesting subclass of starlike functions called uniformly starlike functions denoted by UST and an analogous subclass of convex functions called uniformly convex functions, denoted by UCV. From [5, 7] we have

$$f \in UCV \Leftrightarrow Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|, z \in U.$$

In [7], Ronning introduced a new class S_p of starlike functions which has more manageable properties. The classes UCV and S_p were further extended by Kanas and Wisniowska in [3, 4] as $k - UCV(\alpha)$ and $k - ST(\alpha)$. The classes of uniformly spirallike and uniformly convex spirallike were introduced by Ravichandran et al [6]. This was further generalized in [10] as $UCSP(\alpha, \beta)$. In [11], Herb Silverman introduced the subclass T of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic and univalent in the unit disc U. Motivated by [12], new subclasses with negative coefficients $UCSPT(\alpha,\beta)$ and $SP_pT(\alpha,\beta)$ were introduced and studied in [9]. A function $f(z)$ defined by (1) is in $UCSPT(\alpha,\beta)$ if

$$Re\left\{e^{-i\alpha}\left(1+\frac{zf"(z)}{f'(z)}\right)\right\}\geq\left|\frac{zf"(z)}{f'(z)}\right|+\beta, \tag{2}$$

$|\alpha|<\frac{\pi}{2}, 0\leq\beta<1$. For the class $UCSPT(\alpha,\beta)$, [9] proved the following lemma.

Lemma 1. *A function $f(z)=z-\sum_{n=2}^{\infty}a_nz^n$ is in $UCSPT(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}(2n-\cos\alpha-\beta)n\,a_n\leq\cos\alpha-\beta. \tag{3}$$

Using (1), the functions $f(z)\in UCSPT(\alpha,\beta)$ will satisfy

$$a_2\leq\frac{(cos\alpha-\beta)}{2(4-cos\alpha-\beta)}. \tag{4}$$

Let $UCSPT_c(\alpha,\beta)$ be the class of functions in $UCSPT(\alpha,\beta)$ of the form

$$f(z)=z-\frac{c(\cos\alpha-\beta)z^2}{2(4-\cos\alpha-\beta)}-\sum_{n=3}^{\infty}a_nz^n, \tag{5}$$

$(a_n\geq 0)$, where $0\leq c\leq 1$. When c = 1 we get $UCSPT_1(\alpha,\beta)=UCSPT(\alpha,\beta)$.

2. Coefficient Estimat

Theorem 2. *The function f(z) defined by (5) belongs to $UCSPT_c(\alpha,\beta)$ if and only if*

$$\sum_{n=3}^{\infty}(2n-\cos\alpha-\beta)na_n\leq(1-c)(\cos\alpha-\beta). \tag{6}$$

The result is sharp.

Proof. Taking

$$a_2=\frac{c(\cos\alpha-\beta)}{2(4-\cos\alpha-\beta)}, 0\leq c\leq 1, \tag{7}$$

in (3) we get the required result. Also the result is sharp for the function

$$f(z)=z-\frac{c(\cos\alpha-\beta)z^2}{2(4-\cos\alpha-\beta)}-\frac{(1-c)(\cos\alpha-\beta)z^n}{n(2n-\cos\alpha-\beta)}, (n\geq 3). \tag{8}$$

Corollary 3. *If $f(z)$ defined by (5) is in the class $UCSPT_c(\alpha, \beta)$ then,*

$$a_n \leq \frac{(1-c)(\cos\alpha - \beta)}{n(2n - \cos\alpha - \beta)}, (n \geq 3). \tag{9}$$

The result is sharp for the function $f(z)$ given in (8).

3. Closure Theorems

Theorem 4. *The class $UCSPT_c(\alpha, \beta)$ is closed under convex linear combination.*

Proof. Let $f(z)$ defined by (5) be in $UCSPT_c(\alpha, \beta)$. Now define $g(z)$ by

$$g(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(4 - \cos\alpha - \beta)} - \sum_{n=3}^{\infty} b_n z^n, (b_n \geq 0). \tag{10}$$

If $f(z)$ and $g(z)$ belong to $UCSPT_c(\alpha, \beta)$ then it is enough to prove that the function $H(z)$ defined by

$$H(z) = \lambda f(z) + (1-\lambda)g(z), (0 \leq \lambda \leq 1) \tag{11}$$

is also in $UCSPT_c(\alpha, \beta)$.

$$H(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(4 - \cos\alpha - \beta)} - \sum_{n=3}^{\infty} (\lambda a_n + (1-\lambda)b_n) z^n. \tag{12}$$

Using theorem (2.1) we get

$$\sum_{n=3}^{\infty} (2n - \cos\alpha - \beta) n (\lambda a_n + (1-\lambda)b_n) \leq (1-c)(\cos\alpha - \beta). \tag{13}$$

Hence $H(z)$ is in $UCSPT_c(\alpha, \beta)$. Thus $UCSPT_c(\alpha, \beta)$ is closed under convex linear combination.

Theorem 5. *Let the functions*

$$f_j(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(4 - \cos\alpha - \beta)} - \sum_{n=3}^{\infty} a_{n,j} z^n, (a_{n,j} \geq 0), \tag{14}$$

be in the class $UCSPT_c(\alpha, \beta)$ for every j = 1,2,...m. Then the function $F(z)$ defined by

$$F(z) = \sum_{j=1}^{m} d_j f_j(z), (d_j \geq 0), \tag{15}$$

is also in the same class $UCSPT_c(\alpha, \beta)$ *where*

$$\sum_{j=1}^{m} d_j = 1. \tag{16}$$

Proof. Using (14) and (16) in (15) we have

$$F(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(4 - cos\alpha - \beta)} - \sum_{n=3}^{\infty}\left[\sum_{j=1}^{m} d_j a_n, j\right] z^n. \tag{17}$$

Each $f_j(z) \in UCSPT_c(\alpha, \beta)$ for j = 1,2,...m, theorem (2.1) gives

$$\sum_{n=3}^{\infty}(2n - \cos\alpha - \beta)na_{n,j} \leq (1 - c)(\cos\alpha - \beta), \tag{18}$$

for j =1,2,..m. Hence we get

$$\sum_{n=3}^{\infty} n(2n - \cos\alpha - \beta)\left[\sum_{j=1}^{m} d_j a_{n,j}\right] = \sum_{j=1}^{m} d_j \left[\sum_{n=3}^{\infty} n(2n - \cos\alpha - \beta)a_{n,j}\right] \leq (1 - c)(\cos\alpha - \beta).$$

This implies $F(z) \in UCSPT_c(\alpha, \beta)$, by theorem(2.1).

Theorem 6. *Let*

$$f_2(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(4 - \cos\alpha - \beta)} \tag{19}$$

and

$$f_n(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(4 - \cos\alpha - \beta)} - \frac{(1 - c)(\cos\alpha - \beta)z^n}{n(2n - \cos\alpha - \beta)}, \tag{20}$$

for n = 3,4,.... Then $f(z)$ *is in* $UCSPT_c(\alpha, \beta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z) \tag{21}$$

where $\lambda_n \geq 0$*and* $\sum_{n=2}^{\infty} \lambda_n = 1$.

Proof. First assume that $f(z)$ can be expressed in the form(3.12). Then we have

$$f(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(4 - \cos\alpha - \beta)} - \sum_{n=3}^{\infty} \frac{(1 - c)(\cos\alpha - \beta)}{n(2n - \cos\alpha - \beta)}\lambda_n z^n. \tag{22}$$

But

$$\sum_{n=3}^{\infty}\frac{(1-c)(\cos\alpha-\beta)}{n(2n-\cos\alpha-\beta)}\lambda_n n(2n-\cos\alpha-\beta)=(1-c)(\cos\alpha-\beta)(1-\lambda_2)$$
$$\leq(1-c)(\cos\alpha-\beta). \quad (23)$$

Hence from (2.1) it follows that $f(z)\in UCSPT_c(\alpha,\beta)$. Conversely, we assume that $f(z)$ defined by (1.5) is in the class $UCSPT_c(\alpha,\beta)$. Then by using (2.4), we get

$$a_n\leq\frac{(1-c)(\cos\alpha-\beta)}{n(2n-\cos\alpha-\beta)},(n=3,4,...).$$

Taking $\lambda_n=\frac{n(2n-\cos\alpha-\beta)a_n}{(1-c)(\cos\alpha-\beta)},(n=3,4,...)$ and $\lambda_2=1-\sum_{n=3}^{\infty}\lambda_n$, we have (21). Hence the proof of theorem (6) is complete.

Corollary 7. *The extreme points of the class $UCSPT_c(\alpha,\beta)$ are the functions $f_n(z),(n\geq 2)$ given by theorem (6).*

4. Distortion Theorems

In order to obtain the distortion bounds for the function $f(z)\in UCSPT_c(\alpha,\beta)$, we need the following lemmas.

Lemma 8. *Let the function $f_3(z)$ be defined by*

$$f_3(z)=z-\frac{c(\cos\alpha-\beta)z^2}{2(4-\cos\alpha-\beta)}-\frac{(1-c)(\cos\alpha-\beta)z^3}{3(6-\cos\alpha-\beta)}. \quad (24)$$

Then, for $0\leq r<1$ and $0\leq c\leq 1$,

$$|f_3(re^{i\theta})|\geq r-\frac{c(\cos\alpha-\beta)r^2}{2(4-\cos\alpha-\beta)}-\frac{(1-c)(\cos\alpha-\beta)r^3}{3(6-\cos\alpha-\beta)}, \quad (25)$$

with equality for $\theta=0$. For either $0\leq c<c_0$ and $0\leq r\leq r_0$ or $c_0\leq c\leq 1$,

$$|f_3(re^{i\theta})|\leq r+\frac{c(\cos\alpha-\beta)r^2}{2(4-\cos\alpha-\beta)}-\frac{(1-c)(\cos\alpha-\beta)r^3}{3(6-\cos\alpha-\beta)}, \quad (26)$$

with equality for $\theta=\pi$. Further, for $0\leq c<c_0$ and $r_0\leq r<1$,

$$|f_3(re^{i\theta})|\leq r\Bigg[[1+\frac{9c^2(\cos\alpha-\beta)(6-\cos\alpha-\beta)}{16(1-c)(4-\cos\alpha-\beta)^2}]$$
$$+r^2(\cos\alpha-\beta)[\frac{2(1-c)}{3(6-\cos\alpha-\beta)}-\frac{c^2(\cos\alpha-\beta)}{8(4-\cos\alpha-\beta)^2}]$$
$$+\frac{r^4(1-c)(\cos\alpha-\beta)^2}{(6-\cos\alpha-\beta)}[\frac{(1-c)}{9(6-\cos\alpha-\beta)}+\frac{c^2(cos\alpha-\beta)}{16(4-\cos\alpha-\beta)^2}]\Bigg]^{1/2},$$

with equality for $\theta = \cos^{-1}\left[\frac{c(\cos\alpha-\beta)(1-c)r^2-3c(6-\cos\alpha-\beta)}{8(1-c)(4-\cos\alpha-\beta)r}\right]$, *where*

$$c_0 = \frac{1}{2(\cos\alpha-\beta)}\Bigg[(11\cos\alpha+11\beta-49) + \sqrt{(49-11\cos\alpha-11\cos\beta)^2-32(\cos\alpha-\beta)(4-\cos\alpha-\beta)}\Bigg] \tag{27}$$

and

$$r_0 = \frac{1}{c(1-c)(\cos\alpha-\beta)}\Bigg[-4(1-c)(4-\cos\alpha-\beta) + \sqrt{16(1-c)^2(4-\cos\alpha-\beta)^2+3c^2(1-c)(6-\cos\alpha-\beta)(\cos\alpha-\beta)}\Bigg]. \tag{28}$$

Proof. We employ the techniques used by Silverman and Silvia[12]. Since

$$\frac{\partial|f_3(re^{i\theta})|^2}{\partial\theta} = \frac{(\cos\alpha-\beta)r^3\sin\theta}{(4-\cos\alpha-\beta)}\Bigg[c+\frac{8(1-c)(4-\cos\alpha-\beta)r\cos\theta}{3(6-\cos\alpha-\beta)} - \frac{c(1-c)r^2(\cos\alpha-\beta)}{3(6-\cos\alpha-\beta)}\Bigg], \tag{29}$$

we see that $\frac{\partial|f_3(re^{i\theta})|^2}{\partial\theta} = 0$, for θ_1=0, θ_2=π and

$$\theta_3 = \cos^{-1}\left[\frac{(\cos\alpha-\beta)c(1-c)r^2-3c(6-\cos\alpha-\beta)}{8(1-c)(4-\cos\alpha-\beta)r}\right], \tag{30}$$

since θ_3 is a valid root only when $-1 \le \cos\theta_3 \le 1$. Hence there is a third root if and only if $r_0 \le r < 1$ and $0 \le c \le c_0$. Thus the results of the theorem follow by comparing the extremal values $|f_3(re^{i\theta_k})|$, (k=1,2,3) on the appropriate intervals.

Lemma 9. *Let the function* $f_n(z)$ *be defined by (20) and* $n \ge 4$. *Then*

$$|f_n(re^{i\theta})| \le |f_n(-r)|. \tag{31}$$

Proof. Since $f_n(z) = z - \frac{c(\cos\alpha-\beta)z^2}{2(4-\cos\alpha-\beta)} - \frac{(1-c)(\cos\alpha-\beta)z^n}{n(2n-\cos\alpha-\beta)}$ and $\frac{r^n}{n}$ is a decreasing function of n, we have

$$\begin{aligned} |f_n(re^{i\theta})| &\le r + \frac{c(\cos\alpha-\beta)r^2}{2(4-\cos\alpha-\beta)} + \frac{(1-c)(\cos\alpha-\beta)r^n}{n(2n-\cos\alpha-\beta)} \\ &\le r + \frac{c(\cos\alpha-\beta)r^2}{2(4-\cos\alpha-\beta)} + \frac{(1-c)(\cos\alpha-\beta)r^4}{4(8-\cos\alpha-\beta)} = -f_4(-r), \end{aligned}$$

which gives (31).

Theorem 10. *Let the function $f(z)$ defined by (5) belong to the class $UCSPT_c(\alpha,\beta)$. Then for $0 \le r < 1$,*

$$|f(re^{i\theta})| \ge r - \frac{c(\cos\alpha-\beta)r^2}{2(4-\cos\alpha-\beta)} - \frac{(1-c)(\cos\alpha-\beta)r^3}{3(6-\cos\alpha-\beta)},$$

with equality for $f_3(z)$ at z=r and

$$|f(re^{i\theta})| \le max\{max_\theta|f_3(re^{i\theta})|, -f_4(-r)\},$$

where $max_\theta|f_3(re^{i\theta})|$ is given by lemma 4.1.

The proof is obtained by comparing the bounds of lemma 4.1 and lemma 4.2.

Corollary 11. *Let the function $f(z)$ be defined by (1) be in the class $UCSPT(\alpha,\beta)$. Then for $|z| = r < 1$, we have*

$$r - \frac{(\cos\alpha-\beta)r^2}{2(4-\cos\alpha-\beta)} \le |f(z)| \le r + \frac{(\cos\alpha-\beta)r^2}{2(4-\cos\alpha-\beta)}.$$

The result is sharp.

Corollary 12. *Let the function $f(z)$ be defined by (5) be in the class $UCSPT_c(\alpha,\beta)$. Then the disk $|z| < 1$ is mapped onto a domain that contains the disk*

$$|w| < \frac{6(6-\cos\alpha-\beta)(4-\cos\alpha-\beta)-(\cos\alpha-\beta)(8+10c+(c+2)(\cos\alpha-\beta))}{6(4-\cos\alpha-\beta)(6-\cos\alpha-\beta)}.$$

The result is sharp with the extremal function

$$f_3(z) = z - \frac{c(\cos\alpha-\beta)z^2}{2(4-\cos\alpha-\beta)} - \frac{(1-c)(\cos\alpha-\beta)z^3}{3(6-\cos\alpha-\beta)}.$$

Proof. The result follows by letting $r \to 1$ in theorem 4.3.

Lemma 13. *Let the function $f_3(z)$ be defined by (24) . Then for $0 \le r < 1$ and $0 \le c \le 1$,*

$$|f_3'(re^{i\theta})| \ge 1 - \frac{c(\cos\alpha-\beta)r}{(4-\cos\alpha-\beta)} - \frac{(1-c)(\cos\alpha-\beta)r^2}{(6-\cos\alpha-\beta)},$$

with equality for $\theta = 0$. For either $0 \le c < c_1$ and $o \le r \le r_1$ or $c_1 \le c \le 1$,

$$|f_3'(re^{i\theta})| \le 1 + \frac{c(\cos\alpha-\beta)r}{(4-\cos\alpha-\beta)} - \frac{(1-c)(\cos\alpha-\beta)r^2}{(6-\cos\alpha-\beta)},$$

with equality for $\theta = \pi$. *Further,* $0 \leq c < c_1$ *and* $r_1 \leq r < 1$,

$$|f_3'(re^{i\theta})| \leq \Bigg\{ \left[1 + \frac{c^2(\cos\alpha - \beta)(6 - \cos\alpha - \beta)}{4(1-c)(4 - \cos\alpha - \beta)^2} \right] + (\cos\alpha - \beta)\left[\frac{2(1-c)}{(6 - \cos\alpha - \beta)} + \frac{c^2(\cos\alpha - \beta)}{2(4 - \cos\alpha - \beta)^2} \right] r^2 + \frac{(1-c)(\cos\alpha - \beta)^2}{6 - \cos\alpha - \beta} \left[\frac{(1-c)}{(6 - \cos\alpha - \beta)} + \frac{c^2(\cos\alpha - \beta)}{4(4 - \cos\alpha - \beta)^2} \right] r^4 \Bigg\}^{1/2},$$

with equality for

$$\theta = \cos^{-1}\left[\frac{c(1-c)(\cos\alpha - \beta)r^2 - c(6 - \cos\alpha - \beta)}{4(1-c)r(4 - \cos\alpha - \beta)} \right],$$

where

$$c_1 = \frac{-(22 - 6\cos\alpha - 4\beta) + \sqrt{(22 - 6\cos\alpha - 4\beta)^2 + 16(4 - \cos\alpha - \beta)(\cos\alpha - \beta)}}{2(\cos\alpha - \beta)}$$

and

$$r_1 = \frac{1}{c(1-c)(\cos\alpha - \beta)} \Bigg\{ -2(1-c)(4 - \cos\alpha - \beta) + \sqrt{4(1-c)^2(4 - \cos\alpha - \beta)^2 - c^2(1-c)(\cos\alpha - \beta)(6 - \cos\alpha - \beta)} \Bigg\}.$$

The proof of lemma(4.4) is given in the same way as lemma(4.1).

Theorem 14. *Let the function* $f(z)$ *defined by (1.5) be in the class* $UCSPT_c(\alpha, \beta)$. *Then for* $0 \leq r < 1$,

$$|f'(re^{i\theta})| \geq 1 - \frac{c(\cos\alpha - \beta)r}{(4 - \cos\alpha - \beta)} - \frac{(1-c)(\cos\alpha - \beta)r^2}{(6 - \cos\alpha - \beta)},$$

with equality for $f_3'(z)$ *at z=r and*

$$\left| f'(re^{i\theta}) \right| \leq max\{ max_\theta \left| f_3'(re^{i\theta}) \right|, f_4'(-r) \},$$

where $max_\theta \left| f_3'(re^{i\theta}) \right|$ *is given by lemma (4.4).*

Remark: For c=1 in theorem 6 we obtain:

Corollary 15. *Let the function* $f(z)$ *defined by (1.1) be in the class* $UCSPT(\alpha, \beta)$. *Then for* $|z| = r < 1$, *we have*

$$1 - \frac{(\cos\alpha - \beta)r}{4 - \cos\alpha - \beta} \leq \left| f'(z) \right| \leq 1 + \frac{(\cos\alpha - \beta)r}{4 - \cos\alpha - \beta},$$

the result is sharp.

5. Radii of starlikeness and convexity

Theorem 16. *Let the function $f(z)$ defined by(5) be in the class $UCSPT_c(\alpha,\beta)$. Then $f(z)$ is starlike of order $\rho(0 \le \rho < 1)$ in the disc $|z| < r_1(\alpha,\beta,c,\rho)$ where $r_1(\alpha,\beta,c,\rho)$ is the largest value for which*

$$\frac{c(\cos\alpha-\beta)(2-\rho)r}{2(4-\cos\alpha-\beta)} + \frac{(1-c)(\cos\alpha-\beta)(n-\rho)r^{n-1}}{n(2n-\cos\alpha-\beta)} \le 1-\rho, \tag{32}$$

for $n \ge 3$. The result is sharp with the extremal function

$$f_n(z) = z - \frac{c(\cos\alpha-\beta)z^2}{2(4-\cos\alpha-\beta)} - \frac{(1-c)(\cos\alpha-\beta)z^n}{n(2n-\cos\alpha-\beta)}, \tag{33}$$

for some n.

Proof. It suffices to show that

$$\Big|\frac{zf'(z)}{f(z)} - 1\Big| \le 1-\rho, (o \le \rho < 1),$$

for $|z| < r_1(\alpha,\beta,c,\rho)$. Note that

$$\Big|\frac{zf'(z)}{f(z)} - 1\Big| \le \frac{\frac{c(\cos\alpha-\beta)r}{2(4-\cos\alpha-\beta)} + \sum_{n=3}^{\infty}(n-1)a_n r^{n-1}}{1-\frac{c(\cos\alpha-\beta)r}{2(4-\cos\alpha-\beta)} - \sum_{n=3}^{\infty} a_n r^{n-1}}$$
$$\le 1-\rho,$$

for $|z| \le r$ if and only if

$$\frac{c(\cos\alpha-\beta)(2-\rho)r}{2(4-\cos\alpha-\beta)} + \sum_{n=3}^{\infty}(n-\rho)a_n r^{n-1} \le 1-\rho.$$

Since f(z) is in $UCSPT_c(\alpha,\beta)$ from (2.1) we may take

$$a_n = \frac{(1-c)(\cos\alpha-\beta)\lambda_n}{n(2n-\cos\alpha-\beta)}, (n \ge 3),$$

where $\lambda_n \ge 0 (n \ge 3)$ and $\sum_{n=3}^{\infty}\lambda_n \le 1$. For each fixed r, we choose the positive integer $n_0 = n_0(r)$ for which $\frac{(n-\rho)r^{n-1}}{n}$ is maximal. Then it follows that

$$\sum_{n=3}^{\infty}(n-\rho)a_n r^{n-1} \le \frac{(1-c)(\cos\alpha-\beta)(n_0-\rho)r^{n_0-1}}{n_0(2n_0-\cos\alpha-\beta)}.$$

Hence f(z) is starlike of order ρ in $|z| < r_1(\alpha, \beta, c, \rho)$ provided that

$$\frac{c(\cos\alpha - \beta)(2-\rho)r}{2(4-\cos\alpha-\beta)} + \frac{(1-c)(\cos\alpha-\beta)(n_0-\rho)r^{n_0-1}}{n_0(2n_0-\cos\alpha-\beta)} \leq 1-\rho.$$

We find the value $r_0 = r_0(\alpha, \beta, c, \rho)$ and the corresponding integer $n_0(r_0)$ so that

$$\frac{c(\cos\alpha - \beta)(2-\rho)r_0}{2(4-\cos\alpha-\beta)} + \frac{(1-c)(\cos\alpha-\beta)(n_0-\rho)r_0^{n_0-1}}{n_0(2n_0-\cos\alpha-\beta)} = 1-\rho.$$

Then this value r_0 is the radius of starlikeness of order ρ for functions f(z) belonging to the class $UCSPT_c(\alpha, \beta)$.

We prove the following theorem concerning the radius of convexity of order ρ for functions in the class $UCSPT_c(\alpha, \beta)$.

Theorem 17. *Let the function $f(z)$ be defined by (5) be in the class $UCSPT_c(\alpha, \beta)$. Then $f(z)$ is convex of order $\rho(0 \leq \rho < 1)$ in the disc $|z| < r_2(\alpha, \beta, c, \rho)$, where $r_2(\alpha, \beta, c, \rho)$ is the largest value for which*

$$\frac{c(\cos\alpha - \beta)(2-\rho)r}{(4-\cos\alpha-\beta)} + \frac{(1-c)(\cos\alpha-\beta)(n-\rho)r^{n-1}}{(2n-\cos\alpha-\beta)} \leq 1-\rho,$$

for $n \geq 3$. The result is sharp for the function f(z) given by (33).

6. The class $UCSPT_{c_n,N}(\alpha, \beta)$

We now fix finitely many coefficients instead of fixing just the second coefficients. Let $UCSPT_{c_n,N}(\alpha, \beta)$ denote the class of functions in $UCSPT_c(\alpha, \beta)$ of the form

$$f(z) = z - \sum_{n=2}^{N} \frac{c_n(\cos\alpha-\beta)z^n}{n(2n-\cos\alpha-\beta)} - \sum_{n=N+1}^{\infty} a_n z^n,$$

where $0 \leq \sum_{n=2}^{N} c_n = c \leq 1$. Note that $UCSPT_{c_n,2}(\alpha, \beta) = UCSPT_c(\alpha, \beta)$.

Theorem 18. *The extreme points of the class $UCSPT_{c_n,N}(\alpha, \beta)$ are*

$$z - \sum_{n=2}^{N} \frac{c_n(\cos\alpha-\beta)z^n}{n(2n-\cos\alpha-\beta)}$$

and

$$z - \sum_{n=2}^{N} \frac{c_n(\cos\alpha-\beta)z^n}{n(2n-\cos\alpha-\beta)} - \frac{(1-c)(\cos\alpha-\beta)z^n}{n(2n-\cos\alpha-\beta)},$$

for n=N+1,N+2,....

The characterization of the extreme points enables us to solve the standard extremal problems in the same manner as was done in $UCSPT_c(\alpha, \beta)$. The details are omitted.

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