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MAPPING PROPERTIES OF GENERALIZED Q–INTEGRAL OPERATOR OF P–VALENT FUNCTIONS INVOLVING THE RUSCHEWEYH DERIVATIVE AND THE GENERALIZED SALAGEAN OPERATOR

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Abstract. In this paper, we consider generalized integral operator using certain operator of fractional q–derivative together with Ruscheweyh derivative and generalized Salagean operator. Convexity properties of this operator on some classes of analytic function are determine. By considering the coefficient inequalities of this operator, we obtain some necessary and sufficient conditions, of such operator, on convexity of complex order for certain class of analytic functions involving convex functions of complex order. Furthermore, several corollaries and consequences of the main results are also mentioned.

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1. Introduction and Preliminaries

Due to the boundless potential of demonstrated applications in variously many fields of mathematical science, the fractional calculus operator has expanded their widespread acceptance and significance. In the study of q– theory, the fractional q–calculus seems to be the extension of the ordinary fractional calculus. In recent history, theory of q–calculus operator have been widely applied in many areas of higher mathematical researches, for instance: the problems in optimal control, the ordinary fractional calculus, the problems in optimal control, the ordinary fractional calculus, the problem solving in the q–integral, q–difference equation, q–transform analysis and also in the geometric function theory of complex analysis (see for example [1, 2, 3]).

In the past, mathematicians have studied on various aspects of integral operators. The sufficient conditions of integral operators in order to transform analytic functions in to classes of functions with convexity, starlikeness, and univalent properties are famously focused on (see for example [4, 5, 6, 7, 8, 9, 10]). Recently, Selvakumaran et al. [2] introduced the q–integral operators for certain analytic functions of a unit disc, by using the concept and theory of fractional q–calculus. They also studied on convexity properties of such q–integral operators on some classes of analytic functions which was defined by a linear multiplier fractional q–differintegral operator.

The aim of this paper is to consider a linear multiplier fractional q–differintegral operator which is more generalized than the operator defined formally. We also introduce new classes of analytic functions involving the q–derivative which are comprehension of the classes studied before. In the studying, the mentioned operator on these new classes of analytic functions, we derive the several properties of these q–integral operators of p–valent functions on our new classes. Furthermore, other interesting properties are also discussed.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc of the complex plane, and $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \tag{1}$$

which is analytic in $\mathbb{D}$. Also denote $\mathcal{T}_p$ the subclass of $\mathcal{A}_p$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, \ \ a_{p+k} \geq 0, \ \ (k = 1, 2, ...) \tag{2}$$

In particular, we set $\mathcal{A}_1 \equiv \mathcal{A}$ and $\mathcal{T}_1 \equiv \mathcal{T}$.

A function $f \in \mathcal{A}_p$ is said to be p–valently starlike of complex order b ($b \in \mathbb{C} - \{0\}$) and type γ ($0 \leq \gamma < p$) if f satisfies the condition

$$\Re\left\{p + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - p\right)\right\} > \gamma, \ \ z \in \mathbb{D}.$$

We say that f is in the class $\mathcal{S}_p(b,\gamma)$ for such functions. Denote by $\mathcal{S}(p,b,\gamma)$, the subclass of $\mathcal{S}_p(b,\gamma)$ consisting of function $f \in \mathcal{A}_p$ for which

$$\left|\frac{1}{b}\left(\frac{zf'(z)}{f(z)} - p\right)\right| < p - \gamma, \ \ z \in \mathbb{D}.$$

On the other hand, a function $f \in \mathcal{A}_p$ is said to be convex of complex order b ($b \in \mathbb{C} - \{0\}$) and type γ ($0 \leq \gamma < p$), that is, $f \in \mathcal{K}_p(b,\gamma)$ if it satisfies the following condition

$$\Re\left\{p + \frac{1}{b}\left(1 + \frac{zf''(z)}{f'(z)} - p\right)\right\} > \gamma, \ \ z \in \mathbb{D}.$$

Analogously, we denote the subclass $\mathcal{K}(p,b,\gamma)$ of $\mathcal{K}_p(b,\gamma)$ consisting of functions $f \in \mathcal{A}_p$ for which

$$\left|\frac{1}{b}\left(1+\frac{zf''(z)}{f'(z)}-p\right)\right| < p-\gamma, \quad z \in \mathbb{D}.$$

In particular, for $p=1$, the classes $\mathcal{S}_1(b,\gamma)$ and $\mathcal{K}_1(b,\gamma)$ are respectively the classes $\mathcal{S}(b,\gamma)$ and $\mathcal{K}(b,\gamma)$ that were introduced by Frasin [11]. We note that, for $b=1$, the classes $\mathcal{S}_p(1,\gamma) \equiv \mathcal{S}_p^*(\gamma)$ and $\mathcal{K}_p(1,\gamma) \equiv \mathcal{K}_p(\gamma)$ become the well–known classes, namely, the class of p–valently starlike and p–valently convex functions of order γ, respectively. Also, we set $\mathcal{S}(p,1,\gamma) \equiv \mathcal{S}^*(p,\gamma)$ and $\mathcal{K}(p,1,\gamma) \equiv \mathcal{K}(p,\gamma)$.

Furthermore, a function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{B}_p(\eta,\gamma)$, for a nonnegative number η and $0 \le \gamma < p$, if it satisfies

$$\left|\frac{f'(z)}{z^{p-1}}\left(\frac{z^p}{f(z)}\right)^{\eta} - p\right| < p-\gamma, \quad z \in \mathbb{D}.$$

In particular, for $p=1$, we set $\mathcal{B}_1(\eta,\gamma) \equiv \mathcal{B}(\eta,\gamma)$ which was introduced by Frasin and Jahangiri [12]. The family $\mathcal{B}_p(\eta,\gamma)$ is a comprehensive class of analytic functions which includes various new classes of analytic univalent functions such as $\mathcal{B}(1,\alpha) \equiv \mathcal{S}^*$, and when $\alpha = 2$, the subclass $\mathcal{B}(2,\alpha)$ has been introduced by Frasin and Darus [13].

Fractional q–calculus operators

In the thoery of q–calculus, the q–shifted factorial is defined for $\alpha, q \in \mathbb{C}$, $n \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ as a product of n factors by

$$(\alpha;q)_n = \begin{cases} 1 & , \ n=0 \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}) & , \ n \in \mathbb{N}, \end{cases} \tag{3}$$

and in terms of the basic analogue of the gamma function

$$(q^{\alpha};q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, \ \ (n>0),$$

where the q–gamma function [14, 15] is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty}, \quad (0<q<1).$$

We note that, if $|q| < 1$, the q–shifted factorial (3) remains meaningful for $n = \infty$ as a convergent infinite product:

$$(\alpha; q)_\infty = \prod_{k=0}^{\infty} (1 - \alpha q^k).$$

It is well known that $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1$, where $\Gamma(x)$ is the ordinary Euler gamma fucntion.

In view of the relation

$$\lim_{q \to 1^-} \frac{(q^\alpha; q)_n}{(1-q)^n} = (\alpha)_n,$$

we observe that the q–shifted factorial (3) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$.

Let $\mu \in \mathbb{C}$ be fixed. A set $A \subset \mathbb{C}$ is called a μ–geometric set if for $z \in A$, $\mu z \in A$. Let f be a function defined on a q–geometric set. The Jackson's q–derivative and q–integral of a function on subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [14], pp.19–22)

$$D_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}, \quad (z \neq 0,\ q \neq 0), \tag{4}$$

and

$$\int_0^z f(t) d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f(zq^k). \tag{5}$$

In case $f(z) = z^n$, the q–derivative of $f(z)$, where n is a positive integer, is given by

$$D_q z^n = \frac{z^n - (zq)^n}{(1-q)z} = [n]_q z^{n-1},$$

where

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1},$$

and is called q–analogue of n. As $q \to 1$, we have $[n]_q = 1 + q + \cdots + q^{n-1} \to n$.

We now recall the definition of the fractional q–calculus operators of a complex–valued function $f(z)$, which were recently studied by Purohit and Raina [16].

Definition 1. *(Fractional q–integral operator) The fractional q–integral operator $I^\delta_{q,z}$ of a function $f(z)$ of order δ $(\delta > 0)$ is defined by*

$$I^\delta_{q,z} f(z) = D^{-\delta}_{q,z} f(z) = \frac{1}{\Gamma_q(\delta)} \int_0^z (z - tq)_{1-\delta} f(t) d_q t, \tag{6}$$

where $f(z)$ is analytic in a simply connected region of the z–plane containing the origin. Here, the term $(z-tq)_{\delta-1}$ is a q–binomial function defined by

$$(z-tq)_{\delta-1} = z^{\delta-1}\prod_{k=0}^{\infty}\left[\frac{1-(tq/z)q^k}{1-(tq/z)q^{\delta+k-1}}\right] = z^{\delta}_{1}\Phi_0\left[q^{-\delta+1};-;q,\frac{tq^{\delta}}{z}\right]. \quad (7)$$

According to Gasper and Rahman [14], the series ${}_1\Phi_0\,[\delta;-;q,z]$ is single–valued when $|\arg(z)| < \pi$, therefore, the function $(z-tq)_{\delta-1}$ in (7) is single–valued when $|\arg(-tq^{\delta}/z)| < \pi$, $|tq^{\delta/z}| < 1$ and $|\arg(z)| < \pi$.

Definition 2. *(Fractional q–derivative operator) The fractional q–derivative operator $D^{\delta}_{q,z}$ of a function $f(z)$ of order δ $(0 \le \delta < 1)$ is defined by*

$$D^{\delta}_{q,z}f(z) = D_{q,z}I^{1-\delta}_{q,z}f(z) = \frac{1}{\Gamma_q(1-\delta)}D_q\int_0^z (z-tq)_{-\delta}f(t)d_qt,$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\alpha}$ is removed as in Definition 1 above. In particular, for $\delta = 1$, we have $D^{1}_{q,z}f(z) \equiv D_qf(z)$.

Definition 3. *(Extended fractional q–derivative operator) Under the hypotheses of Definition 2, the fractional q–derivative for a function $f(z)$ of order δ is defined by*

$$D^{\delta}_{q,z}f(z) = D^{m}_{q,z}I^{m-\delta}_{q,z}f(z), \quad (8)$$

where $m-1 \le \delta < m$, $m \in \mathbb{N}_0$.

In addition, we recall the definition of q–differintegral operator $\Omega^{\delta}_{q,p} : \mathcal{A}_p \to \mathcal{A}_p$, for $\delta < p+1$, $0 < q < 1$ and $n \in \mathbb{N}$, as follows:

$$\begin{aligned}\Omega^{\delta}_{q,p}f(z) &= \frac{\Gamma_q(p+1-\delta)}{\Gamma_q(p+1)}z^{\delta}D^{\delta}_{q,z}f(z)\\ &= z^p + \sum_{k=p+1}^{\infty}\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}a_kz^k, \quad (9)\end{aligned}$$

where $D^{\delta}_{q,p}$ in (9) represents, respectively, a fractional q–integral of $f(z)$ of order δ when $-\infty < \delta < 0$ and a fractional q–derivative of $f(z)$ of order δ when $0 \le \delta < p+1$. By (9), It is easy to see that $\Omega^{0}_{q,p}f(z) = f(z)$.

The operator $\mathcal{RD}^{\delta,m}_{q,p,\lambda}$

Next, we recall the generalized AL–Oboudi type differential operator $\mathcal{D}^{\delta,m}_{q,p,\lambda} : \mathcal{A}_p \to \mathcal{A}_p$, for $\lambda \geq 0$ and $m \in \mathbb{N}$, as follows:

$$\begin{aligned}
\mathcal{D}^{\delta,0}_{q,p,\lambda} f(z) &= f(z), \\
\mathcal{D}^{\delta,1}_{q,p,\lambda} f(z) &= (1-\lambda)\Omega^{\delta}_{q,p} f(z) + \frac{\lambda z}{[p]_q} D_q \left(\Omega^{\delta}_{q,p} f(z)\right), \\
\mathcal{D}^{\delta,2}_{q,p,\lambda} f(z) &= \mathcal{D}^{\delta,1}_{q,p,\lambda} \left(\mathcal{D}^{\delta,1}_{q,p,\lambda} f(z)\right), \\
&\vdots \\
\mathcal{D}^{\delta,m}_{q,p,\lambda} f(z) &= \mathcal{D}^{\delta,1}_{q,p,\lambda} \left(\mathcal{D}^{\delta,m-1}_{q,p,\lambda} f(z)\right),
\end{aligned} \tag{10}$$

which was introduce by Selvakumaran et al. [2]. We note that, if $f \in \mathcal{A}_p$ is given by (1), then by (10) we have

$$\mathcal{D}^{\delta,m}_{q,p,\lambda} f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)} \left[1-\lambda+\frac{[k]_q}{[p]_q}\lambda\right]\right)^m a_k z^k. \tag{11}$$

We note that, by specializing the parameters in the operator $\mathcal{D}^{\delta,m}_{q,p,\lambda}$, this operator reduces to many well–known differential operators. For example, when $p = 1$, and $q \to 1$, the operator $\mathcal{D}^{\delta,m}_{q,1,\lambda}$ reduces to the operator introduced by AL–Oboudi and AL–Amoudi [17]. Also, if $\delta = 0$, $p = 1$ and $q \to 1$, it becomes the operator D^n_{λ} introduced by AL–Oboudi [18]. The special case $D^n_0 = D^n$ was considered by Salagean [19].

Analogously, we define the generalized Ruschewyh type differential operator $\mathcal{R}^{\delta,m}_{q,p} : \mathcal{A}_p \to \mathcal{A}_p$, for $m \in \mathbb{N}$, as follows:

$$\begin{aligned}
\mathcal{R}^{\delta,0}_{q,p} f(z) &= f(z), \\
\mathcal{R}^{\delta,1}_{q,p} f(z) &= \frac{z}{[p]_q} D_{q,p} \left(\Omega^{\delta}_{q,p} f(z)\right), \\
2\mathcal{R}^{\delta,2}_{q,p} f(z) &= \frac{z}{[p]_q} D_q \left(\mathcal{R}^{\delta,1}_{q,p} f(z)\right) + \mathcal{R}^{\delta,1}_{q,p} f(z), \\
&\vdots \\
(m+1)\mathcal{R}^{\delta,m+1}_{q,p} f(z) &= \frac{z}{[p]_q} D_q \left(\mathcal{R}^{\delta,m}_{q,p} f(z)\right) + m\mathcal{R}^{\delta,m}_{q,p} f(z).
\end{aligned} \tag{12}$$

For $f \in \mathcal{A}_p$ is given by (1), then by (12) we have

$$\mathcal{R}_{q,p}^{\delta,m} f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)} \right)^m C_m \left(\frac{[k]_q}{[p]_q} \right) a_k z^k, \tag{13}$$

where

$$C_m(t) = \frac{t(t+1)\dots(t+m)}{m!}.$$

We note that, for $\delta = 0$, $p = 1$, and $q \to 0$, the operator $\mathcal{R}_{q,1}^{0,m}$ becomes the operator introduced by Ruscheweyh [20].

Using the operator $\mathcal{D}_{q,p,\lambda}^{\delta,m}$ and $\mathcal{R}_{q,p}^{\delta,m}$, we now define a linear multiplier fractional q–differintegral operator $\mathcal{RD}_{q,p,\lambda}^{\delta,m}$ as follows:

Definition 4. *Let $\alpha, \lambda \geq 0$, $\delta < p+1$, and $m \in \mathbb{N}$. Denote by $\mathcal{RD}_{q,p,\lambda}^{\delta,m}$ the operator given by $\mathcal{RD}_{q,p,\lambda}^{\delta,m} : \mathcal{A}_p \to \mathcal{A}_p$,*

$$\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m} f(z) = (1-\alpha)\mathcal{R}_{q,p}^{\delta,m} f(z) + \alpha \mathcal{D}_{q,p,\lambda}^{\delta,m} f(z). \tag{14}$$

In particular, when $q \to 1$, we set $\mathcal{RD}_{q,p,\lambda}^{\delta,m} = \mathcal{RD}_{p,\lambda}^{\delta,m}$.

Remark 1. *From* (11) *and* (13)*, for $f \in \mathcal{A}_p$ is given by* (1)*, then by* (14) *we conclude that*

$$\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m} f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)} \right)^m \times \left[(1-\alpha) C_m \left(\frac{[k]_q}{[p]_q} \right) + \alpha \left(1 - \lambda + \frac{[k]_q}{[p]_q}\lambda \right)^m \right] a_k z^k. \tag{15}$$

From definition (14), it is easy to see that the operator $\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m}$ generalizes of differential operator $\mathcal{D}_{q,p,\lambda}^{\delta,m}$ in (10) and $\mathcal{R}_{q,p}^{\delta,m}$ in (12). For $p = 1$ and $q \to 1$, we obtain $\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m} = RD_{\gamma,\alpha}^{m}$, the operator introduced by Lupas [21]. The special case of operator $RD_{\gamma,\alpha}^{m}$ are also studied in [22, 23].

Next, we introduce the q–analogues to the classes $\mathcal{S}_{q,p}(b,\gamma)$, $\mathcal{K}_{q,p}(b,\gamma)$ and $\mathcal{B}_{q,p}(\eta,\gamma)$ given as follow.

A function $f \in \mathcal{A}_p$ is said to be p–valently starlike of complex order b ($b \in \mathbb{C} - \{0\}$) and type γ ($0 \leq \gamma < [p]_q$) with respect to q–differentiation if f satisfies the condition

$$\Re \left\{ [p]_q + \frac{1}{b} \left(\frac{z D_q (f(z))}{f(z)} - [p]_q \right) \right\} > \gamma, \quad z \in \mathbb{D}.$$

We say that f is in the class $\mathcal{S}_{q,p}(b,\gamma)$ for such functions. Denote by $\mathcal{S}_q(p,b,\gamma)$, the subclass of $\mathcal{S}_{q,p}(b,\gamma)$ consisting of function $f \in \mathcal{A}_p$ for which

$$\left|\frac{1}{b}\left(\frac{zD_q\left(f(z)\right)}{f(z)} - [p]_q\right)\right| < [p]_q - \gamma, \quad z \in \mathbb{D}.$$

On the other hand, a function $f \in \mathcal{A}_p$ is said to be convex of complex order b ($b \in \mathbb{C} - \{0\}$) and type γ ($0 \le \gamma < [p]_q$) with respect to q–differentiation, that is $f \in \mathcal{K}_{q,p}(b,\gamma)$ if it satisfies the following condition

$$\Re\left\{[p]_q + \frac{1}{b}\left(1 + \frac{zD_q^2\left(f(z)\right)}{D_q\left(f(z)\right)} - [p]_q\right)\right\} > \gamma, \quad z \in \mathbb{D}.$$

Analogously, we denote the subclass $\mathcal{K}_q(p,b,\gamma)$ of $\mathcal{K}_{q,p}(b,\gamma)$ consisting of functions $f \in \mathcal{A}_p$ for which

$$\left|\frac{1}{b}\left(1 + \frac{zD_q^2\left(f(z)\right)}{D_q\left(f(z)\right)} - [p]_q\right)\right| < [p]_q - \gamma, \quad z \in \mathbb{D}.$$

We note that, the special case $\mathcal{S}_{q,p}(1,\gamma) \equiv \mathcal{S}^*_{q,p}(\gamma)$ and $\mathcal{K}_{q,p}(1,\gamma) \equiv \mathcal{K}_{q,p}(\gamma)$ become the classes of starlike and convex functions with respect to q–differentiation which were introduced by Selvakumaran [2]. Also, we set $\mathcal{S}_q(p,1,\gamma) \equiv \mathcal{S}_q(p,\gamma)$ and $\mathcal{K}_q(p,1,\gamma) \equiv \mathcal{K}_q(p,\gamma)$.

Furthermore, a function $f \in \mathcal{A}_p$ is said to be the class $\mathcal{B}_{q,p}(\eta,\gamma)$ if it satisfies

$$\left|\frac{D_q\left(f(z)\right)}{z^{p-1}}\left(\frac{z^p}{f(z)}\right)^{\eta} - [p]_q\right| < [p]_q - \gamma, \quad z \in \mathbb{D},$$

where $\eta \ge 0$ and $0 \le \gamma < [p]_q$. In particular, we see that $\mathcal{B}_{q,p}(1,\gamma) \equiv \mathcal{S}_q(p,\gamma)$.

The Generalized p–Valent q–Integral operator $F_{q,n}$

We now consider the p–valent q–integral operator using the operator $\mathcal{RD}^{\delta,\mu,m}_{q,p,\lambda,\alpha}$ defined (14) as follow:

Definition 5. *Let $m_j \in \mathbb{N}_0$, $\mu_j \in \mathbb{R}^+$ and $f_j \in \mathcal{A}_p$, for all $j = 1, 2, \ldots, n$. Then $F_{q,n} : \mathcal{A}_p^n \to \mathcal{A}_p$ is defined by*

$$\begin{aligned} F_{q,n}(z) &= \mathcal{F}^{\delta,\mu,m}_{q,p,\lambda,\alpha}(f_1, f_2, \ldots, f_n)(z) \\ &= \int_0^z [p]_q t^{p-1} \prod_{j=1}^{n}\left(\frac{\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha} f_j(t)}{t^p}\right)^{\mu_j} d_q t, \end{aligned} \tag{16}$$

In particular, when $q \to 1$, we set $F_{q,n} = F_n$.

It is interesting to observe that several well–known integral operators are special cases of the operators $F_{q,n}$. We list a few of them in the following remarks.

Remark 2. *We note that if $\alpha = 1$, then the integral operator $F_{q,n}$ and becomes the operators $\mathcal{F}^{\delta,\mu,m}_{q,p,\lambda}$ which was introduced by Selvakumaran et al. [2].*

Remark 3. *Letting $p = 1$, $\delta = 0$, $\alpha = 1$, and $q \to 1$, the integral operator $F_{q,n}$ is the integral operator I which was introduced by Bulut [24]. Upon setting $\delta = 0$, $\alpha = 1$, $\lambda = 1$, and $q \to 1$, we obtain operator $F_n = \mathcal{F}_{p,n,l,\mu}$ proposed by Saltik [9]. Moreover, if $\delta = 0$, $\alpha = 0$, and $q \to 1$, the operator F_n becomes $\mathcal{F}_{p,m,l,\mu}$ studied by Deniz et al. [8].*

Remark 4. *Letting $m_j = 0$ for all $j = 1, 2, \ldots, n$ and $q \to 1$, the integral operator F_n is the operator F_p which was introduced Frasin in [4]. The special case of this integral operator has been widely studied by many authors, see [10, 25, 26, 27, 28] for $p = 1$ and [4, 5, 6] for $p \geq 1$.*

In this paper, we obtain convexity properties for the q–integral operator $F_{q,n}$ on the classes $\mathcal{S}_q(p, b, \gamma)$ and $\mathcal{B}_{q,p}(\eta, \gamma)$. Moreover, we obtain some necessary and sufficient conditions for $F_{q,n}$ defined on $\mathcal{T}^n_p$ to be in the class $\mathcal{K}_q(p, b, \zeta)$. As special cases, several corollaries and consequences of the main results are also given.

2. Main Results

Convexity properties on the class $\mathcal{S}_q(p, b, \gamma)$

We begin to derive the convexity of complex order b and type ζ with respect to q–differentiation of the operator $F_{q,n}$ on the class $\mathcal{S}_q(p, b, \gamma)$.

Theorem 1. *For $j = 1, 2, \ldots, n$, let $m_j \in \mathbb{N}_0$, $\mu_j \in \mathbb{R}^+$, $0 \leq \gamma_j < [p]_q$, $\beta \geq 0$, $b \in \mathbb{C} - \{0\}$, and $\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha} f_j \in \mathcal{S}_q(p, b, \gamma_j)$. If*

$$0 \leq \zeta := [p]_q - \frac{(p - [p]_q)}{|b|} \left| 1 - \sum_{j=1}^{n} \mu_j \right| - \sum_{j=1}^{n} \mu_j \left([p]_q - \gamma_j\right), \tag{17}$$

then the q–integral operator $F_{q,n}$ defined in (16) *is in the class $\mathcal{K}_q(p, b, \zeta)$. Furthermore, $F_{q,n} \in \mathcal{K}_{q,p}(b, \zeta)$.*

Proof. From the definition of integral operator in (16), it is easy to see that $F_{q,n} \in \mathcal{A}_p$. By calculating the q–derivative of $F_{q,n}$, we obtain that

$$D_q\left(F_{q,n}(z)\right) = [p]_q z^{p-1} \prod_{j=1}^{n} \left(\frac{\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha} f_j(z)}{z^p} \right)^{\mu_j}.$$

Now, by logarithmic q–differentiation, we have

$$\frac{\ln q}{q-1}\left[\frac{D_q^2\left(F_{q,n}(z)\right)}{D_q\left(F_{q,n}(z)\right)}\right]=\frac{\ln q}{q-1}\left[\frac{p-1}{z}+\sum_{j=1}^{n}\mu_j\left(\frac{D_q\left(\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m_j}f_j(z)\right)}{\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m_j}f_j(z)}-\frac{p}{z}\right)\right].$$

Therefore, by multiplying by z and some calculations give

$$\begin{aligned}1+\frac{zD_q^2\left(F_{q,n}(z)\right)}{D_q\left(F_{q,n}(z)\right)}-[p]_q=&\sum_{j=1}^{n}\mu_j\left(\frac{zD_q\left(\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m_j}f_j(z)\right)}{\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m_j}f_j(z)}-[p]_q\right)\\&+\left(p-[p]_q\right)\left(1-\sum_{j=1}^{n}\mu_j\right).\end{aligned}\tag{18}$$

Then, by multiplying (18) with $1/b$, we obtain that

$$\begin{aligned}\left|\frac{1}{b}\left(1+\frac{zD_q^2\left(F_{q,n}(z)\right)}{D_q\left(F_{q,n}(z)\right)}-[p]_q\right)\right|\leq&\frac{\left(p-[p]_q\right)}{|b|}\left|1-\sum_{j=1}^{n}\mu_j\right|\\&+\sum_{j=1}^{n}\mu_j\left|\frac{1}{b}\left(\frac{zD_q\left(\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m_j}f_j(z)\right)}{\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m_j}f_j(z)}-[p]_q\right)\right|.\end{aligned}\tag{19}$$

Since $\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m_j}f_j\in\mathcal{S}_q(p,b,\gamma_j)$, (19) can be rewritten as

$$\left|\frac{1}{b}\left(1+\frac{zD_q^2\left(F_{q,n}(z)\right)}{D_q\left(F_{q,n}(z)\right)}-[p]_q\right)\right|\leq\frac{\left(p-[p]_q\right)}{|b|}\left|1-\sum_{j=1}^{n}\mu_j\right|+\sum_{j=1}^{n}\mu_j\left([p]_q-\gamma_j\right).$$

Therefore, $F_{q,n}\in\mathcal{K}_q(p,b,\zeta)$. It follows that $F_{q,n}\in\mathcal{K}_{q,p}(b,\zeta)$. Now, the proof is completed.

In Theorem 1, by taking $q\to 1$, we obtain the following.

Corollary 2. *For $j=1,2,\ldots,n$, let $m_j\in\mathbb{N}_0$, $\mu_j\in\mathbb{R}^+$, $0\leq\gamma_j<p$, $b\in\mathbb{C}-\{0\}$, and $\mathcal{RD}_{p,\lambda,\alpha}^{\delta,m_j}f_j\in\mathcal{S}(p,b,\gamma_j)$. If*

$$0\leq\zeta:=p-\sum_{j=1}^{n}\mu_j\left(p-\gamma_j\right),$$

then the integral operator F_n defined in (16) is in the class $\mathcal{K}(p,b,\zeta)$. Furthermore, $F_n\in\mathcal{K}_p(b,\zeta)$.

Also, by taking $b = 1$ in Corollary 2, we obtain the following.

Corollary 3. *For $j = 1, 2, \ldots, n$, let $m_j \in \mathbb{N}_0$, $\mu_j \in \mathbb{R}^+$, $0 \leq \gamma_j < p$, and $\mathcal{RD}^{\delta,m_j}_{p,\lambda,\alpha} f_j \in \mathcal{S}(p, \gamma_j)$. If*

$$0 \leq \zeta := p - \sum_{j=1}^{n} \mu_j \left(p - \gamma_j\right),$$

then the integral operator F_n defined in (16) *is in the class $\mathcal{K}(p, \zeta)$. Furthermore, $F_n \in \mathcal{K}_p(\zeta)$.*

Convextiy properties on the class on $\mathcal{B}_{q,p}(\eta, \gamma)$

In this section, we derive the convexity of complex order b and type ζ with respect to q–differentiation of the operator $F_{q,n}$ on the class $\mathcal{B}_{q,p}(\eta, \gamma)$. In order to establish the next result, the following lemma will be required in our investigation.

Lemma 4. *Let the function f be regular in the disk $\mathbb{D}_R = \{z \in \mathbb{D} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order greater than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \quad , z \in \mathbb{D}_R.$$

The equality holds only if $f(z) = e^{i\theta} \cdot \dfrac{M}{R^m} \cdot z^m$, where θ is a constant.

Then, we derive the following result.

Theorem 5. *For $j = 1, 2, \ldots, n$, let $m_j \in \mathbb{N}_0$, $\mu_j \in \mathbb{R}^+$, $0 \leq \gamma_j < [p]_q$, $M_j \geq 1$, $\eta \geq 0$, $b \in \mathbb{C} - \{0\}$, and $\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha} f_j \in \mathcal{B}_{q,p}(\eta, \gamma_j)$ with $\left|\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha} f_j(z)\right| \leq M_j$. If*

$$0 \leq \zeta := [p]_q - \frac{(p - [p]_q)}{|b|} \left|1 - \sum_{j=1}^{n} \mu_j\right| - \sum_{j=1}^{n} \frac{\mu_j}{|b|} \left[\left(2[p]_q - \gamma_j\right) M^{\eta-1} + [p]_q\right], \tag{20}$$

then the q–integral operator $F_{q,n}$ defined in (16) *is in the class $\mathcal{K}_q(p, b, \zeta)$. Furthermore, $F_{q,n} \in \mathcal{K}_{q,p}(b, \zeta)$.*

Proof. From (19), we have

$$\left|\frac{1}{b}\left(1+\frac{zD_q^2\left(F_{q,n}(z)\right)}{D_q\left(F_{q,n}(z)\right)}-[p]_q\right)\right| \le \frac{(p-[p]_q)}{|b|}\left|1-\sum_{j=1}^{n}\mu_j\right| + \sum_{j=1}^{n}\mu_j\left|\frac{1}{b}\left(\frac{zD_q\left(\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j(z)\right)}{\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j(z)}-[p]_q\right)\right|. \tag{21}$$

Consider the last term of above inequality, we see that

$$\begin{aligned}\left|\frac{zD_q\left(\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j(z)\right)}{\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j(z)}-[p]_q\right| &\le \left|\frac{zD_q\left(\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j(z)\right)}{\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j(z)}\right|+[p]_q \\ &\le \left|\frac{D_q\left(\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j(z)\right)}{z^{p-1}}\left(\frac{z^p}{\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j(z)}\right)^{\eta}\right| \\ &\quad\times\left|\frac{\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j(z)}{z^p}\right|^{\eta-1}+[p]_q. \end{aligned}\tag{22}$$

Since $\left|\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j(z)\right| \le M_j$ for all $j=1,2,\ldots,n$, by applying the general Schwarz Lemma 4, we have

$$\left|\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j(z)\right| \le M_j|z|^p. \tag{23}$$

Therefore, from (22), (23), and $\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j \in \mathcal{B}_{q,p}(\beta,\gamma_j)$, we obtain that

$$\begin{aligned}\left|\frac{zD_q\left(\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j(z)\right)}{\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j(z)}-[p]_q\right| &\le \left|\frac{D_q\left(\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j(z)\right)}{z^{p-1}}\left(\frac{z^p}{\mathcal{RD}^{\delta,m_j}_{q,p,\lambda,\alpha}f_j(z)}\right)^{\eta}\right|M^{\eta-1}+[p]_q \\ &\le \left(2[p]_q-\gamma_j\right)M_j^{\eta-1}+[p]_q. \end{aligned}\tag{24}$$

From (21)–(24), we see that

$$\left|\frac{1}{b}\left(1+\frac{zD_q^2\left(F_{q,n}(z)\right)}{D_q\left(F_{q,n}(z)\right)}-[p]_q\right)\right| \le \frac{(p-[p]_q)}{|b|}\left|1-\sum_{j=1}^{n}\mu_j\right| + \sum_{j=1}^{n}\frac{\mu_j}{|b|}\left[\left(2[p]_q-\gamma_j\right)M_j^{\eta-1}+[p]_q\right].$$

Therefore, $F_{q,n} \in \mathcal{K}_q(p,b,\zeta)$. It follows that $F_{q,n} \in \mathcal{K}_{q,p}(b,\zeta)$. Now, the proof is completed.

In Theorem 5, by taking $q \to 1$, we obtain the following.

Corollary 6. *For $j = 1,2,\ldots,n$, let $m_j \in \mathbb{N}_0$, $\mu_j \in \mathbb{R}^+$, $0 \le \gamma_j < p$, $M_j \ge 1$, $\eta \ge 0$, $b \in \mathbb{C} - \{0\}$, and $\mathcal{RD}^{\delta,m_j}_{p,\lambda,\alpha} f_j \in \mathcal{B}_p(\eta,\gamma_j)$ with $\left|\mathcal{RD}^{\delta,m_j}_{p,\lambda,\alpha} f_j(z)\right| \le M_j$. If*

$$0 \le \zeta := p - \sum_{j=1}^{n} \frac{\mu_j}{|b|} \left[(2p-\gamma_j) M_j^{\eta-1} + p\right],$$

then the integral operator F_n defined in (16) is in the class $\mathcal{K}(p,b,\zeta)$. Furthermore, $F_n \in \mathcal{K}_p(b,\zeta)$.

Also, by taking $b = 1$ in Corollary6, we obtain the following.

Corollary 7. *For $j = 1,2,\ldots,n$, let $m_j \in \mathbb{N}_0$, $\mu_j \in \mathbb{R}^+$, $0 \le \gamma_j < p$, $M_j \ge 1$, $\eta \ge 0$, and $\mathcal{RD}^{\delta,m_j}_{p,\lambda,\alpha} f_j \in \mathcal{B}_p(\eta,\gamma_j)$ with $\left|\mathcal{RD}^{\delta,m_j}_{p,\lambda,\alpha} f_j(z)\right| \le M_j$. If*

$$0 \le \zeta := p - \sum_{j=1}^{n} \mu_j \left[(2p-\gamma_j) M_j^{\eta-1} + p\right],$$

then the integral operator F_n defined in (16) is in the class $\mathcal{K}(p,\zeta)$. Furthermore, $F_n \in \mathcal{K}_p(\zeta)$.

Coefficient inequality of the integral operator $F_{q,n}$ on the class $\mathcal{T}_p^n$

In this section, we derive the necessary and sufficient conditions for the integral operator $F_{q,n}$ of the from (16) on the class $\mathcal{T}_p^n$ in order to be the class $\mathcal{K}_q(p,b,\gamma)$. Before deriving the following results, we set $F_{q,n} \equiv \mathcal{F}^{\delta,\mu,m}_{q,p,\lambda,\alpha}(f_1,f_2,\ldots,f_n)$, where $f_j \in \mathcal{T}_p$ of the from

$$f_j(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,j} z^k, \quad a_{k,j} \ge 0, \;\; k \ge p+1, \; j = 1,2,3,\ldots,n.$$

In particular, for $n = 1$, we set $F_{q,1} = F_q$ and $f_1(z) = f(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k$.

Theorem 8. *Let the function $f_j \in \mathcal{T}_p$ for $j = 1, 2, \ldots, n$. If the integral operator $F_{q,n} \in \mathcal{K}_q(p, b, \gamma)$, then*

$$\sum_{j=1}^{n} \mu_j \left(\frac{\sum\limits_{k=p+1}^{\infty} \left([k]_q - [p]_q\right) \Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k) a_{k,j}}{1 - \sum\limits_{k=p+1}^{\infty} \Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k) a_{k,j}} \right) \leq C_{q,p}^{n}(\mu, \gamma, b), \tag{25}$$

where

$$C_{q,p}^{n}(\mu, \gamma, b) = (p - [p]_q) \left| 1 - \sum_{j=1}^{n} \mu_j \right| + |b| \left([p]_q - \gamma\right), \tag{26}$$

and

$$\Psi_{q,p,\lambda,\alpha}^{\delta,m}(k) = \left(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)} \right)^m \left[(1-\alpha) C_m \left(\frac{[k]_q}{[p]_q} \right) + \alpha \left(1 - \lambda + \frac{[k]_q}{[p]_q} \lambda \right)^m \right]. \tag{27}$$

Moreover, the converse also holds if $\sum\limits_{j=1}^{n} \mu_j = 1$.

Proof. Before embarking on the proof, from (18) we recall that

$$\begin{aligned} 1 + \frac{z D_q^2 \left(F_{q,n}(z)\right)}{D_q \left(F_{q,n}(z)\right)} - [p]_q = & \sum_{j=1}^{n} \mu_j \left(\frac{z D_q \left(\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m_j} f_j(z) \right)}{\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m_j} f_j(z)} - [p]_q \right) \\ & + (p - [p]_q) \left(1 - \sum_{j=1}^{n} \mu_j \right). \end{aligned} \tag{28}$$

By (15) and (27), for $f_j \in \mathcal{T}_p$, we get

$$\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m_j} f_j(z) = z^p - \sum_{k=p+1}^{\infty} \Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k) a_{k,j} z^k.$$

From (28), we see that

$$\begin{aligned}
1+&\frac{zD_q^2\left(F_{q,n}(z)\right)}{D_q\left(F_{q,n}(z)\right)}-[p]_q\\
&=\sum_{j=1}^{n}\mu_j\left(\frac{zD_q\left(\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m_j}f_j(z)\right)}{\mathcal{RD}_{q,p,\lambda,\alpha}^{\delta,m_j}f_j(z)}-[p]_q\right)+(p-[p]_q)\left(1-\sum_{j=1}^{n}\mu_j\right)\\
&=\sum_{j=1}^{n}\mu_j\left(\frac{[p]_q z^p-\sum\limits_{k=p+1}^{\infty}[k]_q\Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k)a_{k,j}z^k}{z^p-\sum\limits_{k=p+1}^{\infty}\Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k)a_{k,j}z^k}-[p]_q\right)+(p-[p]_q)\left(1-\sum_{j=1}^{n}\mu_j\right)\\
&=-\sum_{j=1}^{n}\mu_j\left(\frac{\sum\limits_{k=p+1}^{\infty}\left([k]_q-[p]_q\right)\Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k)a_{k,j}z^{k-p}}{1-\sum\limits_{k=p+1}^{\infty}\Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k)a_{k,j}z^{k-p}}\right)+(p-[p]_q)\left(1-\sum_{j=1}^{n}\mu_j\right).
\end{aligned} \tag{29}$$

Now, let $F_{q,n}$ belong to the class $\mathcal{K}_q(p,b,\gamma)$ then

$$\left|1+\frac{zD_q^2\left(F_{q,n}(z)\right)}{D_q\left(F_{q,n}(z)\right)}-[p]_q\right|\le|b|\left([p]_q-\gamma\right). \tag{30}$$

From (26), (29), and (30), we obtain the following inequality:

$$\left|\sum_{j=1}^{n}\mu_j\left(\frac{\sum\limits_{k=p+1}^{\infty}\left([k]_q-[p]_q\right)\Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k)a_{k,j}z^{k-p}}{1-\sum\limits_{k=p+1}^{\infty}\Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k)a_{k,j}z^{k-p}}\right)\right|\le C_{q,p}^{n}(\mu,\gamma,b). \tag{31}$$

Since $\Re\{z\}\le|z|$ and putting $z=r$ $(0\le r<1)$, we have

$$\Re\left\{\sum_{j=1}^{n}\mu_j\left(\frac{\sum\limits_{k=p+1}^{\infty}\left([k]_q-[p]_q\right)\Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k)a_{k,j}r^{k-p}}{1-\sum\limits_{k=p+1}^{\infty}\Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k)a_{k,j}r^{k-p}}\right)\right\}\le C_{q,p}^{n}(\mu,\gamma,b). \tag{32}$$

By letting $r \to 1^-$ along the real axis, (32) leads us to the desired assertion (25) of Theorem 8.

Conversely, by applying (25), (29), and the assumption $\sum_{j=1}^{n} \mu_j = 1$, we can see that

$$\begin{aligned}
\left|1 + \frac{zD_q^2\left(F_{q,n}(z)\right)}{D_q\left(F_{q,n}(z)\right)} - [p]_q\right| &\leq \sum_{j=1}^{n} \mu_j \left| \frac{\sum_{k=p+1}^{\infty} \left([k]_q - [p]_q\right) \Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k) a_{k,j} z^{k-p}}{1 - \sum_{k=p+1}^{\infty} \Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k) a_{k,j} z^{k-p}} \right| \\
&\leq \sum_{j=1}^{n} \mu_j \left(\frac{\sum_{k=p+1}^{\infty} \left([k]_q - [p]_q\right) \Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k) a_{k,j} |z|^{k-p}}{1 - \sum_{k=p+1}^{\infty} \Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k) a_{k,j} |z|^{k-p}} \right) \\
&\leq \sum_{j=1}^{n} \mu_j \left(\frac{\sum_{k=p+1}^{\infty} \left([k]_q - [p]_q\right) \Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k) a_{k,j}}{1 - \sum_{k=p+1}^{\infty} \Psi_{q,p,\lambda,\alpha}^{\delta,m_j}(k) a_{k,j}} \right) \\
&\leq |b| \left([p]_q - \gamma\right).
\end{aligned}$$

Hence, by (30), we infer that $F_{q,n} \in \mathcal{K}_{q,}(p, b, \gamma)$. Now, the proof is completed.

For $n = 1$, $\mu_1 = \mu$, and $m_1 = m$ in Theorem 8, we obtain the following.

Theorem 9. *Let the function $f \in \mathcal{T}_p$. If the integral operator $F_q \in \mathcal{K}_{q,}(p, b, \gamma)$ then*

$$\sum_{k=p+1}^{\infty} \left[\mu\left([k]_q - [p]_q\right) + C_{q,p}(\mu, \gamma, b)\right] \Psi_{q,p,\lambda,\alpha}^{\delta,m}(k) a_k \leq C_{q,p}(\mu, \gamma, b), \tag{33}$$

where $C_{q,p}^1(\mu, \gamma, b) = C_{q,p}(\mu, \gamma, b)$ and $\Psi_{q,p,\lambda,\alpha}^{\delta,m}(k)$ is defined in (27). Moreover, the converse also holds if $\mu = 1$.

Proof. Suppose that the integral operator $F_q \in \mathcal{K}_{q,}(p, b, \gamma)$. Then by (31), we have

$$\mu \left| \frac{\sum_{k=p+1}^{\infty} ([k]_q - [p]_q)\, \Psi_{q,p,\lambda,\alpha}^{\delta,m}(k) a_k z^{k-p}}{1 - \sum_{k=p+1}^{\infty} \Psi_{q,p,\lambda,\alpha}^{\delta,m}(k) a_k z^{k-p}} \right| \leq C_{q,p}(\mu,\gamma,b). \tag{34}$$

Thus, putting $z = r$ $(0 \leq r < 1)$, we obtain

$$\frac{\mu \sum_{k=p+1}^{\infty} ([k]_q - [p]_q)\, \Psi_{q,p,\lambda,\alpha}^{\delta,m}(k) a_k r^{k-p}}{1 - \sum_{k=p+1}^{\infty} \Psi_{q,p,\lambda,\alpha}^{\delta,m}(k) a_k r^{k-p}} \leq C_{q,p}(\mu,\gamma,b). \tag{35}$$

We observe that the expression in denominator on the left–hand side of (35) is positive for $r = 0$ and also for all r $(0 \leq r < 1)$. Then, we obtain that

$$\sum_{k=p+1}^{\infty} [\mu\,([k]_q - [p]_q) + C_{q,p}(\mu,\gamma,b)]\, \Psi_{q,p,\lambda,\alpha}^{\delta,m}(k) a_k r^{k-p} \leq C_{q,p}(\mu,\gamma,b). \tag{36}$$

By letting $r \to 1^-$ along the real axis, (36) leads us to the desired assertion (33) of Theorem 9. Similarly to the proof of Theorem 9, the converse also holds for $\mu = 1$.

Corollary 10. *Let the function $f \in \mathcal{T}_p$. If the integral operator $F_q \in \mathcal{K}_{q,}(p,b,\gamma)$ then*

$$a_k \leq \frac{C_{q,p}(\mu,\gamma,b)}{[\mu\,([k]_q - [p]_q) + C_{q,p}(\mu,\gamma,b)]\, \Psi_{q,p,\lambda,\alpha}^{\delta,m}(k)}, \tag{37}$$

where $C_{q,p}^{1}(\mu,\gamma,b) = C_{q,p}(\mu,\gamma,b)$ and $\Psi_{q,p,\lambda,\alpha}^{\delta,m}(k)$ is defined in (27).

In view of Theorem 8 and Theorem 9, by taking $q \to 1$ then $(p - [p]_q)\left(1 - \sum_{j=1}^{n} \mu_j\right) \to 0$. So, we obtain the following results:

Theorem 11. *Let the function $f_j \in \mathcal{T}_p$ for $j = 1, 2, \ldots, n$. Then, the integral operator $F_n \in \mathcal{K}(p,b,\gamma)$ if and only if*

$$\sum_{j=1}^{n} \mu_j \left(\frac{\sum_{k=p+1}^{\infty} (k-p)\, \Psi_{p,\lambda,\alpha}^{\delta,m_j}(k) a_{k,j}}{1 - \sum_{k=p+1}^{\infty} \Psi_{p,\lambda,\alpha}^{\delta,m_j}(k) a_{k,j}} \right) \leq |b|\,(p-\gamma)\,,$$

where

$$\Psi^{\delta,m}_{p,\lambda,\alpha}(k) = \left(\frac{\Gamma(p+1-\delta)\Gamma(k+1)}{\Gamma(p+1)\Gamma(k+1-\delta)}\right)^m \left[(1-\alpha)C_m\left(\frac{k}{p}\right) + \alpha\left(1-\lambda+\frac{k}{p}\lambda\right)^m\right]. \tag{38}$$

Theorem 12. *Let the function $f \in \mathcal{T}_p$. Then, the integral operator $F \in \mathcal{K}(p,b,\gamma)$ if and only if*

$$\sum_{k=p+1}^{\infty} \left[\mu\left(k-p\right) + |b|\left(p-\gamma\right)\right] \Psi^{\delta,m}_{p,\lambda,\alpha}(k) a_k \leq |b|\left(p-\gamma\right), \tag{39}$$

where $\Psi^{\delta,m}_{p,\lambda,\alpha}(k)$ is defined in (38).

Corollary 13. *Let the function $f \in \mathcal{T}_p$. If the integral operator $F \in \mathcal{K}(p,b,\gamma)$ then*

$$a_k \leq \frac{|b|\left(p-\gamma\right)}{\left[\mu\left(k-p\right) + |b|\left(p-\gamma\right)\right] \Psi^{\delta,m}_{p,\lambda,\alpha}(k)}, \tag{40}$$

where $\Psi^{\delta,m}_{p,\lambda,\alpha}(k)$ is defined in (38).

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