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INTEGRALS INVOLVING $\bar{H}$ - FUNCTION

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Abstract. The present paper deals with various integral formulas involving $\bar{H}$ -function due to Inayat-Hussain multiplied with algebraic functions and special functions.

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1. Introduction

In an attempt to evaluate certain Feynman integral in two different ways which arise in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions. Inayat-Hussain ([4], p.4126) introduced a generalization of Fox's H-function in the form:

$$\bar{H}(z) = \bar{H}^{m,n}_{p,q} = \bar{H}^{m,n}_{p,q}\left[z\middle|\begin{matrix}(\alpha_j, A_j; a_j)_{1,n}, (\alpha_j, A_j)_{n+1,p}\\ (\beta_j, B_j)_{1,m}, (\beta_j, B_j; b_j)_{m+1,q}\end{matrix}\right] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\psi(s)z^s ds, \tag{1}$$

where

$$\psi(s) = \frac{\prod_{j=1}^{m}\Gamma(\beta_j - B_j s)\prod_{j=1}^{n}\{\Gamma(1-\alpha_j + A_j s)\}^{a_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-\beta_j + B_j s)\}^{b_j}\prod_{j=n+1}^{p}\Gamma(\alpha_j - A_j s)} \tag{2}$$

which contains fractional powers of some of the Γ -functions. Here z may be real or complex but is not equal to zero and an empty product is interpreted as unity; p, q, m and n are integers such that $1 \le m \le q, 0 \le n \le p; A_j > 0\,(j = 1, ..., p), B_j > 0\,(j = 1, ..., q)$ and $a_j\,(j = 1, ..., n)$ and $b_j\,(j = m+1, ..., q)$ can be take on non- integer values. The poles of the integrand of (1) are assumed to be simple. The contour in (1) is presumed to be the imaginary axis $\Re(s) = 0$, which is suitable indented in order to avoid the singularities of the gamma functions and to keep these singularities at appropriate sides. It has been shown by Buschman and Srivastava ([13], p.

4708) that the sufficient condition for absolute convergence of the contour integral (1) is given by

$$\Omega=\sum_{j=1}^{m}|B_j|+\sum_{j=1}^{n}|a_jA_j|-\sum_{j=m+1}^{q}|b_jB_j|-\sum_{j=n+1}^{p}|A_j|>0. \tag{3}$$

This condition provides exponential decay of the integrand in (1), and region of absolute convergence of (1) is

$$|\arg z|<\frac{1}{2}\pi\Omega. \tag{4}$$

For further details about the $\bar{H}$ -function the reader is referred to the original paper of Bushman and Srivastava [13] and Inayat-Hussain [4].
When the exponents $a_j=b_j=1,\forall i,j$, then $\bar{H}$ -function reduces to the familiar Fox's H-function defined by Fox [8]; see also Mathai and Saxena [6]:

$$\bar{H}_{p,q}^{m,n}(z)=H_{p,q}^{m,n}\left[z\middle|\begin{array}{c}(\alpha_1,A_1),...,(\alpha_p,A_p)\\(\beta_1,B_1),...,(\beta_q,B_q)\end{array}\right]==\frac{1}{2\pi i}\int_L\chi(s)z^s ds, \tag{5}$$

where

$$\chi(s)=\frac{\prod_{j=1}^{m}\Gamma(\beta_j-B_js)\prod_{j=1}^{n}\Gamma(1-\alpha_j+A_js)}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j+B_js)\prod_{j=n+1}^{p}\Gamma(\alpha_j-A_js)} \tag{6}$$

an empty product is interpreted as unity; the integer m, n, p, q satisfy the inequalities $0\le n\le p$ and $1\le m\le q$, the coefficients $A_j>0(j=1,...,p)$ and $B_j>0(j=1,...,q)$ and the complex parameters α_j and β_j are such that the poles of the integrand are simple and L is a suitable contour of Mellin-Barnes type in complex s-plane separating the poles of $\Gamma(1-\alpha_j+A_js)$ for $j=1,...,n$. The integral in (1) converge absolutely and defines the H- function, analytic in the sector

$$|\arg z|<\frac{1}{2}\pi\lambda^*, \tag{7}$$

where

$$\lambda^*=\sum_{j=1}^{m}B_j-\sum_{j=1}^{q}B_j+\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{p}A_j>0 \tag{8}$$

the point $z=0$ being tacitly excluded.

A detailed account of the H -function is available from the monograph of Mathai and Saxena [6]. Existence conditions, analytic continuation and asymptotic expansions of the H -function have been discussed by Braaksma [7].

A relation connecting $L^{\nu}(z)$, the polytogacithm of complex order ν and the $\bar{H}$ -function is the following :

$$L^{\nu}(z) = H^{1,1:\nu}_{1,2:\nu-1}\left[-z\Big|\begin{matrix}(1,1;\nu)\\(0,1)\,(0,1:\nu-1)\end{matrix}\right], \tag{9}$$

which readily follows on comparing their contour integral definitions. An account of $L^{\nu}(z)$, the polylogarithm of complex order ν is available from the book by Marichev [12].

Existence condition for the $\bar{H}$ -function can be established by following the procedure adopted by Braaksma ([7], pp. 278-279), that the function $\bar{H}(z)$ makes sense and defines an analytic function of z in the following two cases.
I. $\mu > 0$, and $0 < |z| < \infty$ where

$$\mu = \sum_{j=1}^{m}|B_j| + \sum_{j=m+1}^{q}|B_j\beta_j| = \sum_{j=1}^{n}|A_j\alpha_j| - \sum_{j=n+1}^{p}|A_j|. \tag{10}$$

II. $\mu = 0$ and $0 < |z| < \tau^{-1}$ holds.

$$\tau = \{\prod_{j=1}^{m}(B_j)^{-B_j}\}\{\prod_{j=1}^{n}(A_j)^{A_j\alpha_j}\}\{\prod_{j=n+1}^{p}(A_j)^{A_j}\}\{\prod_{j=n+1}^{q}(B_j)^{-B_j\beta_j}\}. \tag{11}$$

By calculating the residue at the poles of $\Gamma(\beta_j - B_j s)$ for $j = 1, ..., m$ in (1) we obtain the following representation of the $\bar{H}$ -function in a computable from as

$$\bar{H}(z) = \bar{H}^{m,n}_{p,q}[z] = \sum_{h=1}^{m}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\psi(\zeta)z^{\zeta}}{\nu! B_h}, \tag{12}$$

where $\zeta = \frac{(\beta_h+v)}{B_h}$ (3) exist for $0 < |z| < \infty$, if $\mu < 0$ or $\mu = 0$ and $0 < |z| < \tau^{-1}$, where τ^{-1} is defined in (11), μ in (10), $\psi(.)$ in (2); $B_h(\beta_j + v_1) \neq B_j(\beta_h + v_2)$ for $j \neq h$, $h = 1, ..., m; v_1, v_2 = 0, 1, 2, ...$.

The behaviour of the $\bar{H}$ -function for small value of $|z|$ follows easily from a result given by Rathai [5]

$$\bar{H}^{P,Q}_{M,N}[z] = O\left(|z|^{\alpha}\right) \tag{13}$$

$$\alpha = \min_{0\leq j\leq M}\Re\left(\frac{b_j}{\beta_j}\right), |z| \to 0. \tag{14}$$

Rathie [5. pg. 303, Eq. 5.4] is also represent $\bar{H}$ in the following form

$$\bar{H}^{m,n}_{p,q}[z] = \bar{I}^{m,n}_{p,q}\left[\begin{matrix}(a_1,\alpha_1,A_1)\dots(a_n,\alpha_n,A_n),(a_{n+1},\alpha_{n+1},1),(a_p,\alpha_p,1)\\(b_1,\beta_1,1)\dots(b_m,\beta_m,1),(b_{m+1},\beta_{m+1},B_{m+1})\dots(b_q,\beta_q,B_q)\end{matrix}\right] \tag{15}$$

$$H_{p,q}^{m,n}\left[z|\begin{array}{c}(a_j,\alpha_j)_p\\(b_j,\beta_j)_q\end{array}\right]=I_{p,q}^{m,n}\left[z|\begin{array}{c}(a_j,\alpha_j,1)_p\\(b_j,\beta_j,1)_q\end{array}\right] \tag{16}$$

$$G_{p,q}^{m,n}\left[z|\begin{array}{c}{}_1(a_j)_p\\{}_1(b_j)_q\end{array}\right]=I_{p,q}^{m,n}\left[z|\begin{array}{c}{}_1(a_j,1,1)_p\\{}_1(b_j,1,1)_q\end{array}\right]. \tag{17}$$

Here $G_{p,q}^{m,n}$ in the G-function, see Luke[16].

2. Integral with Algebraic Function

In this section we will be calculate the $\bar{H}$ -function with some algebraic function.

$$\begin{aligned}I_1&=\int_0^1 y^{-\rho}(1-y)^{\rho-\sigma-1}\bar{H}_{p,q}^{m,n}[zy]dy\\&=\frac{1}{2\pi i}\int_L\psi(s)z^s\{\int_0^1 y^{-\rho+s+1-1}(1-y)^{\rho-\sigma-1}dy\}ds\\&=\frac{1}{2\pi i}\int_L\psi(s)z^s\{B(1-\rho+s,\rho-\sigma)\}ds\\&=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^m\Gamma(b_j-B_js)\prod_{j=1}^n\{\Gamma(1-a_j+A_js)\}^{\alpha_j}}{\prod_{j=m+1}^q\{\Gamma(1-b_j+B_js)\}^{\beta_j}\prod_{j=n+1}^p\Gamma(a_j-A_js)}\frac{\Gamma(1-\rho+s)\Gamma(\rho-\sigma)}{\Gamma(1+s-\sigma)}z^sds\end{aligned}$$

$$=\Gamma(\rho-\sigma)\bar{H}_{p+1,q+1}^{m,n+1}\left[z|\begin{array}{c}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(\rho,1;1)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(\sigma,1;1)\end{array}\right]. \tag{18}$$

$$\begin{aligned}I_2&=\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}\bar{H}_{p,q}^{m,n}(zx)dx\\&=\frac{1}{2\pi i}\int_L\psi(s)z^s\{\int_0^1 x^{\rho+s-1}(1-x)^{\sigma-1}dx\}ds\\&=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^m\Gamma(b_j-B_js)\prod_{j=1}^n\{\Gamma(1-a_j+A_js)\}^{\alpha_j}}{\prod_{j=m+1}^q\{\Gamma(1-b_j+B_js)\}^{\beta_j}\prod_{j=n+1}^p\Gamma(a_j-A_js)}\frac{\Gamma(\rho+s)\Gamma(\rho)}{\Gamma(\rho+\sigma+s)}z^sds\end{aligned}$$

$$=\Gamma(\sigma)\bar{H}_{p+1,q+1}^{m,n+1}\left[z|\begin{array}{c}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(1-\rho,1;1)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(1-\rho-\sigma,1;1)\end{array}\right]. \tag{19}$$

$$\begin{aligned}I_3&=\int_1^\infty x^{-\rho}(x-1)^{\sigma-1}\bar{H}_{p,q}^{m,n}(zx)dx\\&=\frac{1}{2\pi i}\int_L\psi(s)z^s\{\int_1^\infty x^{-\rho+s}(x-1)^{\sigma-1}dx\}ds.\end{aligned}$$

Putting $x = t + 1$ and $dx = dt$ we get

$$= \frac{1}{2\pi i}\int_L \psi(s)z^s\{\int_0^\infty t^{\sigma-1}(t+1)^{-(\sigma+\rho-s-\sigma)}dt\}ds$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(b_j - B_j s)\prod_{j=1}^n\{\Gamma(1-a_j+A_j s)\}^{\alpha_j}}{\prod_{j=m+1}^q\{\Gamma(1-b_j+B_j s)\}^{\beta_j}\prod_{j=n+1}^p\Gamma(a_j - A_j s)}\frac{\Gamma(\rho)\Gamma(\rho-s-\sigma)}{\Gamma(\rho-s)}z^s ds$$

$$= \Gamma(\sigma)\,\bar{H}_{p+1,q+1}^{m+1,n}\left[z\middle|\begin{array}{c}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(\rho;1)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(\rho-\sigma;1)\end{array}\right]. \tag{20}$$

$$I_4 = \int_0^\infty x^{\rho-1}(x+\beta)^{-\sigma}\bar{H}_{p,q}^{m,n}(zx)dx$$

$$= \frac{1}{2\pi i}\int_L \psi(s)z^s\beta^{-\sigma}\{\int_0^\infty x^{\rho+s-1}\left(\frac{x}{\beta}+1\right)^{-\sigma}dx\}ds.$$

Putting $x = t\beta$ and $dx = \beta dt$ we get

$$= \frac{1}{2\pi i}\beta^{\rho-\sigma}\int_L \psi(s)(z\beta)^s\{\int_0^\infty t^{\rho+s-1}(t+1)^{-(\rho+s+\sigma-\rho-s)}dt\}ds$$

$$= \frac{1}{2\pi i}\beta^{\rho-\sigma}\int_L \frac{\prod_{j=1}^m \Gamma(b_j - B_j s)\prod_{j=1}^n\{\Gamma(1-a_j+A_j s)\}^{\alpha_j}}{\prod_{j=m+1}^q\{\Gamma(1-b_j+B_j s)\}^{\beta_j}\prod_{j=n+1}^p\Gamma(a_j - A_j s)}$$

$$\times\frac{\Gamma(\rho+s)\Gamma(\sigma-\rho-s)}{\Gamma(\rho)}(z\beta)^s ds.$$

$$= \frac{\beta^{\rho-\sigma}}{\Gamma(\sigma)}\bar{H}_{p+1,q+1}^{m+1,n+1}\left[z\beta\middle|\begin{array}{c}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(1-\rho,1;1)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(\sigma-\rho,1)\end{array}\right]. \tag{21}$$

$$I_5 = \int_{-1}^1 (1-x)^\rho(1+x)^\sigma\bar{H}_{p,q}^{m,n}(z(1-x)^\mu)dx$$

$$= \frac{1}{2\pi i}\int_L \psi(s)z^s\{\int_{-1}^1(1-x)^{1+\rho+\mu s-1}(1+x)^{1+\sigma-1}dx\}ds. \tag{22}$$

Using the formula([9], p. 261)

$$\int_{-1}^1(1-x)^{n+\alpha}(1+x)^{n+\beta}dx = 2^{2n+\alpha+\beta+1}B(1+\alpha+n,1+\beta+n). \tag{23}$$

Hence (22) becomes

$$= \frac{1}{2\pi i}2^{\rho+\sigma+}\Gamma(1+\sigma)\int_L \frac{\prod_{j=1}^m \Gamma(b_j - B_j s)\prod_{j=1}^n\{\Gamma(1-a_j+A_j s)\}^{\alpha_j}}{\prod_{j=m+1}^q\{\Gamma(1-b_j+B_j s)\}^{\beta_j}\prod_{j=n+1}^p\Gamma(a_j - A_j s)}$$

$$\times \frac{\Gamma\left(1+\rho+\mu s\right)}{\Gamma\left(2+\rho+\mu s+\sigma\right)}\left(2^{\mu}z\right)^{s}ds$$

$$= 2^{\rho+\sigma+1}\Gamma(1+\sigma)\bar{H}^{m,n+1}_{p+1,q+1}\left[2^{\mu}z\Big| \begin{matrix} (a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(-\rho,\mu;1) \\ (b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(-1-\rho-\sigma,\mu;1) \end{matrix}\right]. \tag{24}$$

3. Integral with Jacobi Polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ ([9], p. 254) may be defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_2F_1\left[\begin{matrix} -n, 1+\alpha+\beta+n; & \frac{1-x}{2} \\ 1+\alpha; & \end{matrix}\right] \tag{25}$$

when $\alpha=\beta=0$. The polynomial in (25) becomes the Legendre polynomial ([9], p. 157).

From (25) it follows that $P_n^{(\alpha,\beta)(x)}$ is a polynomial degree precisely n and that

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!}.$$

In dealing with the Jacobi polynomial it is natural to make much use of our knowledge of the ${}_2F_1$ function ([9], p. 45).

$$I_6 = \int_{-1}^{+1} x^{\lambda}\left(1-x\right)^{\alpha}\left(1+x\right)^{\delta}P_n^{(\alpha,\beta)}(x)\bar{H}^{m,n}_{p,q}\left(z\left(1+x\right)^{h}\right)dx$$

$$= \frac{1}{2\pi i}\int_L \psi(s)z^s\{\int_{-1}^{1} x^{\lambda}\left(1-x\right)^{\alpha}\left(1+x\right)^{\delta+hs}P_n^{(\alpha,\beta)}(x)dx\}ds.$$

But by the formula ([14], p. 52)

$$\int_{-1}^{1} x^{\lambda}\left(1-x\right)^{\alpha}\left(1+x\right)^{\delta}P_n^{(\alpha,\beta)}(x)dx$$

$$= (-1)^n\frac{2^{\alpha+\delta+1}\Gamma\left(\delta+1\right)\Gamma\left(n+\alpha+1\right)\Gamma\left(\delta+\beta+1\right)}{n!\Gamma\left(\delta+\beta+n+1\right)\Gamma\left(\delta+\alpha+n+2\right)}$$

$$\times {}_3F_2\left[\begin{matrix} -\lambda, \delta+\beta+1, \delta+1 \\ \delta+\beta+n+1, \delta+\alpha+n+2 \end{matrix} ;1\right].$$

Provided: $\alpha > -1$ and $\beta > -1$. We have

$$= \frac{1}{2\pi i}\int_L \psi(s)z^s(-1)^n 2^{\alpha+\delta+hs+1}$$

$$\times \frac{\Gamma(\delta+hs+1)\Gamma(n+\alpha+1)\Gamma(\delta+hs+\beta+1)}{n!\Gamma(\delta+hs+\beta+n+1)\Gamma(\delta+hs+\alpha+n+2)}$$

$$\times {}_3F_2\left[\begin{matrix}-\lambda,\delta+hs+\beta+1,\delta+hs+1\\ \delta+hs+\beta+n+1,\delta+hs+\alpha+n+2\end{matrix};1\right]$$

$$=\frac{(-1)^n 2^{\alpha+\delta+1}\Gamma(n+\alpha+1)}{n!}\sum_{k=0}^{\infty}\frac{(-\lambda)_k(1)^k}{k!}\frac{1}{2\pi i}$$

$$\times\int_L \frac{\prod_{j=1}^m\Gamma(b_j-B_js)\prod_{j=1}^n\{\Gamma(1-a_j+A_js)\}^{\alpha_j}}{\prod_{j=m+1}^q\{\Gamma(1-b_j+B_js)\}^{\beta_j}\prod_{j=n+1}^p\Gamma(a_j-A_js)}$$

$$\times\frac{\Gamma(\delta+hs+\beta+1+k)\Gamma(\delta+hs+1+k)}{\Gamma(\delta+hs+\beta+n+1+k)\Gamma(\delta+hs+\alpha+n+2+k)}\left(2^hz\right)^s ds$$

$$=\frac{(-1)^n 2^{\alpha+\delta+1}\Gamma(n+\alpha+1)}{n!}\sum_{k=0}^{\infty}\frac{(-\lambda)_k(1)^k}{k!}$$

$$\times\bar{H}^{m,n+2}_{p+2,q+2}\left[2^hz\left|\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(-\delta-\beta-k,h;1),(-k-\delta,h;1)\\(a_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(-\beta-n-k-\delta,h;1)\,(-1-\delta-\alpha-h-k,h;1)\end{matrix}\right.\right]. \tag{26}$$

Provided:
(i) $Re(\lambda)>-1$ and $|\arg z|<\frac{1}{2}\pi\Omega$
(ii) $\alpha>-1,\beta>-1$.

$$I_7=\int_{-1}^{+1}(1-x)^{\delta}(1+x)^{\nu}P_n^{(\mu,\nu)}(x)P_m^{(\rho,\sigma)}(x)\bar{H}^{m,n}_{p,q}\left(z(1-x)^h\right)dx$$

$$=\frac{1}{2\pi i}\int_L\psi(s)z^s\{\int_{-1}^{1}(1-x)^{\delta+hs}(1+x)^{\nu}P_n^{(\mu,\nu)}(x)P_n^{(\rho,\sigma)}(x)dx\}ds.$$

Now using the definition Jacboi polynomial (25)

$$=\frac{(1+\rho)_m}{m!}\sum_{k=0}^{\infty}\frac{(-m)_k(1+\rho+\sigma+m)_k}{(1+\rho)_k 2^k k!}$$

$$\times\frac{1}{2\pi i}\int_L\psi(s)z^s\{\int_{-1}^{1}(1-x)^{\delta+hs+k}(1+x)^{\nu}P_n^{(\mu,\nu)}(x)dx\}ds. \tag{27}$$

Again using (25) in (27) we get

$$=\frac{\Gamma(1+\rho+m)\Gamma(1+\mu+n)}{\Gamma(1+\mu)m!n!}$$

$$\times \sum_{k=0}^{\infty} \frac{(-m)_k\,(-n)_k\,(1+\rho+\sigma+m)_k\,(1+\mu+\nu+n)_k}{2^{2k}\,(k!)^2\,\Gamma\,(1+\rho+k)\,\Gamma\,(1+\mu+k)}$$

$$\times \frac{1}{2\pi i}\int_L \psi(s) 2^{hs} z^s \{\int_{-1}^{+1} (1-x)^{\delta+hs+2k}\,(1+x)^{\nu}\,dx\} ds. \tag{28}$$

By using the formula (23), equation (28) becomes

$$= 2^{\nu+\delta+1}\frac{\Gamma\,(1+\rho+m)\,\Gamma\,(1+\mu+n)}{m!n!}$$

$$\times \sum_{k=0}^{\infty} \frac{(-m)_k\,(-n)_k\,(1+\rho+\sigma+m)_k\,(1+\mu+\nu+n)_k}{(k!)^2\,\Gamma\,(1+\rho+k)\,\Gamma\,(1+\mu+k)}$$

$$\times \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\,(b_j-B_j s)\prod_{j=1}^{n}\{\Gamma\,(1-a_j+A_j s)\}^{\alpha_j}}{\prod_{j=m+1}^{q}\{\Gamma\,(1-b_j+B_j s)\}^{\beta_j}\prod_{j=n+1}^{p}\Gamma\,(a_j-A_j s)}$$

$$\times \frac{\Gamma\,(1+\delta+hs+2k)\,\Gamma\,(1+\nu)}{\Gamma\,(2+\delta+hs+2k+\nu)}\left(2^h z\right)^s ds.$$

$$= \frac{2^{\nu+\delta+1}\Gamma\,(1+\rho+m)\,\Gamma\,(1+\mu+n)}{m!n!}$$

$$\times \sum_{k=0}^{\infty} \frac{(-m)_k\,(-n)_k\,(1+\rho+\sigma+m)_k\,(1+\mu+\nu+n)_k\,\Gamma\,(1+\nu)}{(k!)^2\,\Gamma\,(1+\rho+k)\,\Gamma\,(1+\mu+k)}$$

$$\times \bar{H}_{p+1,q+1}^{m,n+1}\left[2^h z \left| \begin{matrix} (a_j,A_j;\alpha_j)_{1,n}\,,(a_j,A_j)_{n+1,p}\,,(-\delta-2k,h;1) \\ (b_j,B_j)_{1,m}\,,(b_j,B_j;\beta_j)_{m+1,q}\,,(-1-\delta-2k-\nu,h;1) \end{matrix}\right.\right]. \tag{29}$$

Provided: $Re(\nu) > -1$, $|\arg z| < \frac{1}{2}\pi\Omega$ and δ are positive.

$$I_8 = \int_{-1}^{+1} (1-x)^{\rho}\,(1+x)^{\sigma}\,P_n^{(\mu,\nu)}(x)\bar{H}_{p,q}^{m,n}\left(z\,(1-x)^h\,(1+x)^t\right)dx$$

$$= \frac{1}{2\pi i}\int_L \psi(s) z^s \{\int_{-1}^{1} (1-x)^{\rho}\,(1+x)^{\sigma}\,P_n^{(\mu,\nu)}(x)\,(1-x)^{hs}\,(1+x)^{ts}\,dx\} ds.$$

$$= \frac{1}{2\pi i}\int_L \psi(s) z^s \{\int_{-1}^{1} (1-x)^{\rho+hs}\,(1+x)^{\sigma+ts}\,\frac{(1+\mu)_n}{n!}\,{}_2F_1\left[\begin{matrix} -n,1+\mu+\nu+n \\ 1+\mu \end{matrix};\frac{1-x}{2}\right]dx\} ds.$$

$$= \frac{(1+\mu)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k\,(1+\mu+\nu+n)_k}{(1+\mu)_k\,k!2^k}\frac{1}{2\pi i}\int_L \psi(s) z^s$$

$$\times \{\int_{-1}^{1} (1-x)^{\rho+hs+k-1+1}\,(1+x)^{\sigma+ts-1+1}\,dx\} ds. \tag{30}$$

Now using (23) in (30) we get

$$= \frac{(1+\mu)_n}{n!} 2^{\rho+\sigma+1} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\mu+\nu+n)_k}{(1+\mu)_k k!}$$

$$\times \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - B_j s) \prod_{j=1}^{n} \{\Gamma(1-a_j+A_j s)\}^{\alpha_j}}{\prod_{j=m+1}^{q} \{\Gamma(1-b_j+B_j s)\}^{\beta_j} \prod_{j=n+1}^{p} \Gamma(a_j - A_j s)}$$

$$\times \frac{\Gamma(1+k+\rho+hs)\Gamma(1+\sigma+ts)}{\Gamma(k+\rho+hs+\sigma+ts+2)} \left(2^{h+t} z\right)^s ds.$$

$$= \frac{(1+\mu)_n}{n!} 2^{\rho+\sigma+1} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\mu+\nu+n)_k}{k! (1+\mu)_k}$$

$$\times \bar{H}_{p+2,q+1}^{m,n+2} \left[2^{h+t} z \middle| \begin{matrix} (a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (-\rho-k, h; 1), (-\sigma, t; 1) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, (-1-\rho-k-\sigma, h+t; 1) \end{matrix} \right]. \quad (31)$$

Provided: $|\arg z| < \frac{1}{2}\pi\Omega$ and $\Re(\mu) > -1$, and $\Re(\nu) > -1$.

$$I_9 = \int_{-1}^{+1} (1-x)^{\rho} (1+x)^{\sigma} P_n^{(\alpha,\beta)}(x) \bar{H}_{p,q}^{m,n} \left(z(1+x)^{-h}\right) dx$$

$$= \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\alpha+\beta+n)_k}{(1+\alpha)_k k! 2^k} \frac{1}{2\pi i} \int_L \psi(s) z^s$$

$$\times \{\int_{-1}^{1} (1-x)^{\rho+k-1+1} (1+x)^{\sigma-hs-1+1} dx\} ds. \quad (32)$$

Now using (23) in (31) we get

$$= 2^{\rho+\sigma+1} \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\alpha+\beta+n)_k}{(1+\alpha)_k k!}$$

$$\times \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - B_j s) \prod_{j=1}^{n} \{\Gamma(1-a_j+A_j s)\}^{\alpha_j}}{\prod_{j=m+1}^{q} \{\Gamma(1-b_j+B_j s)\}^{\beta_j} \prod_{j=n+1}^{p} \Gamma(a_j - A_j s)}$$

$$\times \frac{\Gamma(1+\sigma-hs)\Gamma(1+\rho+k)}{\Gamma(2++\rho+k+\sigma-hs)} \left(2^{-h} z\right)^s ds.$$

$$= 2^{\rho+\sigma+1} \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\alpha+\beta+n)_k \Gamma(1+\rho+k)}{k! (1+\alpha)_k}$$

$$\times \bar{H}^{m+1,n}_{p+1,q+1}\left[2^{-h}z\left|\begin{array}{c}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(1+\rho,h)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(2+\rho+k+\sigma,h)\end{array}\right.\right]. \quad (33)$$

Provided:
(i) $|\arg z| < \frac{1}{2}\pi\Omega$ and $\Re(\alpha) > -1$, and $\Re(\beta) > -1$
(ii) $\Re\left[\rho + h\min\left(\frac{b_j}{\beta_j}\right)\right] > -1$ where $j = 1, 2, 3, ..., m$.

$$I_{10} = \int_{-1}^{+1} (1-x)^{\rho}(1+x)^{\sigma} P_n^{(\mu,\nu)}(x)\bar{H}^{m,n}_{p,q}\left(z(1-x)^h(1+x)^{-t}\right)dx.$$

$$= \frac{1}{2\pi i}\int_L \psi(s)z^s\{\int_{-1}^{1}(1-x)^{\rho+hs}(1+x)^{\sigma-ts}\frac{(1+\mu)_n}{n!}{}_2F_1\left[\begin{array}{c}-n,1+\mu+\nu+n\\1+\mu\end{array};\frac{1-x}{2}\right]dx\}ds.$$

$$= \frac{(1+\mu)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\mu+\nu+n)_k}{(1+\mu)_k k!2^k}$$

$$\times \frac{1}{2\pi i}\int_L \psi(s)z^s\{\int_{-1}^{1}(1-x)^{\rho+hs+k}(1+x)^{\sigma-ts}dx\}ds. \quad (34)$$

Using (23) in (34), we get

$$= \frac{(1+\mu)_n}{n!}2^{\rho+\sigma+1}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\mu+\nu+n)_k}{(1+\mu)_k k!}$$

$$\times\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-B_js)\prod_{j=1}^{n}\{\Gamma(1-a_j+A_js)\}^{\alpha_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-b_j+B_js)\}^{\beta_j}\prod_{j=n+1}^{p}\Gamma(a_j-A_js)}$$

$$\times\frac{\Gamma(1+\rho+k+hs)\Gamma(1+\sigma-ts)}{\Gamma(k+\rho+hs+\sigma-ts+2)}\left(2^{h-t}z\right)^s ds.$$

$$= 2^{\rho+\sigma+1}\frac{(1+\mu)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\mu+\nu+n)_k}{k!(1+\mu)_k}$$

$$\times\bar{H}^{m+1,n+1}_{p+1,q+2}\left[2^{h-t}z\left|\begin{array}{c}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(-\rho-k,h;1)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(1+\sigma,t)(-1-\rho-k-\sigma,h-t;1)\end{array}\right.\right]. \quad (35)$$

Provided:
(i) $|\arg z| < \frac{1}{2}\pi\Omega$
(ii) $\Re\left[\rho + h\min\left(\frac{b_j}{\beta_j}\right)\right] > -1$ and $\Re\left[\sigma + t\min\left(\frac{b_j}{\beta_j}\right)\right] > -1$ where $j = 1, 2, 3, ..., m$.

4. Special Cases

(i) If we replace δ by $\lambda - 1$ and put $\mu = \nu = \rho = \sigma = 0$ the integral I_7 transform to the following integral involving product of Legendre Polynomials

$$I_{11} = 2^{\lambda} \sum_{k=0}^{\infty} \frac{(-m)_k (-n)_k (1+m)_k (1+n)_k}{(k!)^2 \Gamma(1+k) \Gamma(1+k)}$$

$$\times \bar{H}_{p+1,q+1}^{m,n+1} \left[2^h z \left| \begin{matrix} (a_j, A_j; \alpha_j)_{1,p}, (-\lambda + 1 - 2k, h; 1) \\ (b_j, B_j; \beta_j)_{1,q}, (-\lambda - 2k, h; 1) \end{matrix} \right. \right]. \tag{36}$$

Provided: $|\arg z| < \frac{1}{2}\pi\Omega$.

(ii) If $\mu = \nu = 0, \rho$ is replaced by $\rho - 1$ and σ by $\sigma - 1$ then the integral I_8 transforms into the following integral involving Legendre polynomials

$$I_{12} = \int_{-1}^{+1} (1-x)^{\rho-1} (1+x)^{\sigma-1} P_n(x) \bar{H}_{p,q}^{m,n} \left(z (1-x)^h (1+x)^t \right) dx$$

$$= 2^{\rho+\sigma-1} \sum_{k=0}^{\infty} \frac{(-n)_k (1+n)_k}{k!}$$

$$\times \bar{H}_{p+2,q+1}^{m,n+2} \left[2^{h+t} z \left| \begin{matrix} (a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (-\rho + 1 - k, h; 1), (-\sigma + 1, t; 1) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, (1 - \rho - k - \sigma, h + t; 1) \end{matrix} \right. \right]. \tag{37}$$

Provided: $|\arg z| < \frac{1}{2}\pi\Omega$.

(iii) Replacing ρ by $\rho - 1$, σ by $\sigma - 1$ and putting $\mu = \nu = 0$ the integral I_{10} takes the forms of the following integral

$$I_{13} = \int_{-1}^{+1} (1-x)^{\rho-1} (1+x)^{\sigma-1} P_n(x) \bar{H}_{p,q}^{m,n} \left(z (1-x)^h (1+x)^{-t} \right) dx.$$

$$= 2^{\rho+\sigma-1} \sum_{k=0}^{\infty} \frac{(-n)_k (1+n)_k}{(k!)^2}$$

$$\times \bar{H}_{p+1,q+2}^{m+1,n+1} \left[2^{h-t} z \left| \begin{matrix} (a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (-\rho + 1 - k, h; 1) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, (\sigma, t), (1 - \rho - k - \sigma, h - t; 1) \end{matrix} \right. \right]. \tag{38}$$

Provided:
$|\arg z| < \frac{1}{2}\pi\Omega$, $\Re\left[\rho + h \min\left(\frac{b_j}{\beta_j}\right)\right] > -1$ and $\Re\left[\sigma + t \min\left(\frac{b_j}{\beta_j}\right)\right] > -1$ where $j = 1, 2, 3, ..., m$.

5. Integral With Bessel Maitland Function

The special cases of the Wright function ([2], vol. 3, section 18.1) and ([10], [11]) in the form

$$\phi(B,b;z) \equiv {}_0\psi_1\left[\begin{matrix} - \\ (b,B) \end{matrix} |z\right] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(Bk+b)} \frac{z^k}{k!} \tag{39}$$

with complex $z, b \in C$ and real $B \in R$. When $B = \delta, b = \nu + 1$ and z is replaced by $-z$, the function $\phi(\delta, \nu+1; -z)$ is defined by $J_\nu^\delta(z)$

$$J_\nu^\delta(z) \equiv \phi(\delta, \nu+1 : z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\delta k + \nu + 1)} \frac{(-z)^k}{k!} \tag{40}$$

and such a function is known as the Bessel Maitland function or the Wright generalized Bessel function see ([15], p. 352).

$$I_{14} = \int_0^\infty x^l J_\nu^\tau \bar{H}_{p,q}^{m,n}(zx^\gamma)\,dx$$

$$= \frac{1}{2\pi i}\int_L \Psi(s) z^s \{\int_0^\infty x^{l+\gamma s} J_\nu^\tau(x)dx\}ds.$$

Now using the formula ([14], p. 55)

$$\int_0^\infty x^l J_\nu^\tau(x)dx = \frac{\Gamma(l+1)}{\Gamma(1+\nu-\tau-\tau l)}$$

$\Re(l) > -1, 0 < \tau < 1$.

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(b_j - B_j s) \prod_{j=1}^n \{\Gamma(1-a_j+A_j s)\}^{\alpha_j}}{\prod_{j=m+1}^q \{\Gamma(1-b_j+B_j s)\}^{\beta_j} \prod_{j=n+1}^p \Gamma(a_j - A_j s)}$$

$$\times \frac{\Gamma(l+\gamma s+1)}{\Gamma(1+\nu-\tau-\tau l-\tau\gamma s)} z^s ds$$

$$= \bar{H}_{p+2,q}^{m,n+1}\left[z\left| \begin{matrix} (a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (-l, \gamma; 1), (1+\nu-\tau-\tau l, \tau\gamma) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q} \end{matrix}\right.\right]. \tag{41}$$

Provided:
(i) $|\arg z| < \frac{1}{2}\pi\Omega$
(ii) $\gamma - \tau\gamma > 0, \gamma > 0$
(iii) $0 < \tau < 1$ and $\Re(l+1) > 0$.

6. Integral With Legendre Function

The Legendre functions are solution of Legendre's differential equation ([3], sec. 3.1, vol. 1)

$$\left(1-z^2\right)\frac{d^2\omega}{dz^2}-2z\frac{d\omega}{dz}+\left[\nu\left(\nu+1\right)-\mu^2\left(1-z^2\right)^{-1}\right]\omega=0, \tag{42}$$

where z,ν,μ unrestricted.

Under the substitution $\omega=\left(z^2-1\right)^{\frac{1}{2\mu}}\nu$, equation (42) becomes

$$\left(1-z^2\right)\frac{d^2\nu}{dz^2}-2\left(\mu+1\right)z\frac{d\nu}{dz}+\left(\mu-\nu\right)\left(\nu+\mu+1\right)\nu=0, \tag{43}$$

and with $\xi=\frac{1}{2}-\frac{1}{2}z$ as the independent variable this differential equation becomes

$$\xi\left(1-\xi\right)\frac{d^2\nu}{d\xi^2}+\left(\mu+1\right)\left(1-2\xi\right)\frac{d\nu}{d\xi}+\left(\nu-\mu\right)\left(\nu+\mu+1\right)\nu=0. \tag{44}$$

This is the Gauss hypergeometric type equation with $a=\mu-\nu, b=\nu+\mu+1$ and $c=\mu+1$.

Hence it follows that the function

$$\omega=P_\nu^\mu(z)=\frac{1}{\Gamma\left(1-\mu\right)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2\mu}}.F\left[-\nu,\nu+1;1-\mu;\frac{1}{2}-\frac{1}{2}z\right],|1-z|<2$$

is a solution of (42).

The function $P_\nu^\mu(z)$ is known as the Legendre function of first kind ([3], vol. 1). It is one valued and regular in z-plane supposed cut along the real axis form 1 to $-\infty$.

$$\begin{aligned} I_{15}&=\int_0^1 x^{\sigma-1}\left(1-x^2\right)^{\frac{\delta}{2}}P_\nu^\delta(x)\bar{H}_{m,n}^{p,q}\left(zx^\gamma\right)dx \\ &=\frac{1}{2\pi i}\int_L\psi(s)z^s\{\int_0^1 x^{\sigma-1+\gamma s}\left(1-x^2\right)^{\frac{\delta}{2}}P_\nu^\delta(x)dx\}ds. \end{aligned} \tag{45}$$

Now using the formula ([3], sec 3.12, vol. 1)

$$\int_0^1 x^{\sigma-1}\left(1-x^2\right)^{\frac{\delta}{2}}P_\nu^\delta(x)dx=\frac{\left(-1\right)^\delta\pi^{\frac{1}{2}}2^{-\sigma-\delta}\Gamma\left(\sigma\right)\Gamma\left(1+\delta+\nu\right)}{\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}+\frac{\delta}{2}-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma}{2}+\frac{\delta}{2}+\frac{\nu}{2}\right)\Gamma\left(1-\delta+\nu\right)}. \tag{46}$$

Provided: $\Re(\sigma)>0, \delta=1,2,3,...$.

Now integral (45) becomes

$$=2^{-\sigma-\delta}(-1)^\delta(\pi)^{\frac{1}{2}}\frac{\Gamma\left(1+\delta+\nu\right)}{\Gamma\left(1-\delta+\nu\right)}$$

$$\times \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j - B_j s\right)\prod_{j=1}^{n}\{\Gamma\left(1-a_j+A_j s\right)\}^{\alpha_j}}{\prod_{j=m+1}^{q}\{\Gamma\left(1-b_j+B_j s\right)\}^{\beta_j}\prod_{j=n+1}^{p}\Gamma\left(a_j - A_j s\right)}$$

$$\times \frac{\Gamma\left(\sigma+\gamma s\right)}{\Gamma\left(\frac{1}{2}+\frac{\sigma+\gamma s}{2}+\frac{\delta}{2}-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma+\gamma s}{2}+\frac{\delta}{2}+\frac{\nu}{2}\right)} z^s 2^{-\gamma s} ds$$

$$= 2^{-\sigma-\delta}(-1)^{\delta}(\pi)^{\frac{1}{2}}\frac{\Gamma\left(1+\delta+\nu\right)}{\Gamma\left(1-\delta+\nu\right)}$$

$$\times \bar{H}_{p+1,q+2}^{m,n+1}\left[\frac{z}{2^{\gamma}}\Big| \begin{matrix} \left(a_j,A_j;\alpha_j\right)_{1,n},\left(a_j,A_j\right)_{n+1,p},\left(1-\sigma,\gamma;1\right) \\ \left(b_j,B_j\right)_{1,m},\left(b_j,B_j;\beta_j\right)_{m+1,q},\left(\frac{1}{2}-\frac{\delta}{2}+\frac{\nu}{2}-\frac{\sigma}{2},\frac{\gamma}{2};1\right),\left(-\frac{\sigma}{2}-\frac{\delta}{2}-\frac{\nu}{2},\frac{\gamma}{2};1\right)\end{matrix}\right]. \quad (47)$$

Provided: $|\arg(z)| < \frac{1}{2}\pi\Omega, \sigma > 0$ and δ is non negative integer.

$$I_{16} = \int_0^1 x^{\sigma-1}\left(1-x^2\right)^{\frac{-\delta}{2}} P_{\nu}^{\delta}(x)\bar{H}_{m,n}^{p,q}\left(zx^{\gamma}\right)dx$$

$$= \frac{1}{2\pi i}\int_L \psi(s) z^s\{\int_0^1 x^{\sigma-1+\gamma s}\left(1-x^2\right)^{\frac{-\delta}{2}} P_{\nu}^{\delta}(x)dx\}ds. \quad (48)$$

Now using the formula ([3], sec 3.12, vol. 1)

$$\int_0^1 x^{\sigma-1}\left(1-x^2\right)^{\frac{-\delta}{2}} P_{\nu}^{\delta}(x)dx = \frac{\pi^{\frac{1}{2}}2^{-\sigma+\delta}\Gamma\left(\sigma\right)}{\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}-\frac{\delta}{2}-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma}{2}-\frac{\delta}{2}-\frac{\nu}{2}\right)}. \quad (49)$$

Provided: $\Re(\sigma) > 0, \delta = 1, 2, 3, \ldots$.

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j - B_j s\right)\prod_{j=1}^{n}\{\Gamma\left(1-a_j+A_j s\right)\}^{\alpha_j}}{\prod_{j=m+1}^{q}\{\Gamma\left(1-b_j+B_j s\right)\}^{\beta_j}\prod_{j=n+1}^{p}\Gamma\left(a_j - A_j s\right)}$$

$$\times \frac{2^{-\sigma-\delta}\pi^{\frac{1}{2}}\Gamma\left(\sigma+\gamma s\right)}{\Gamma\left(\frac{1}{2}+\frac{\sigma+\gamma s}{2}-\frac{\delta}{2}-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma+\gamma s}{2}-\frac{\delta}{2}-\frac{\nu}{2}\right)} z^s 2^{-\gamma s} ds$$

$$= 2^{-\sigma-\delta}\pi^{\frac{1}{2}}$$

$$\times \bar{H}_{p+1,q+2}^{m,n+1}\left[\frac{z}{2^{\gamma}}\Big| \begin{matrix} \left(a_j,A_j;\alpha_j\right)_{1,n},\left(a_j,A_j\right)_{n+1,p},\left(1-\sigma,\gamma;1\right) \\ \left(b_j,B_j\right)_{1,m},\left(b_j,B_j;\beta_j\right)_{m+1,q},\left(\frac{1}{2}+\frac{\delta}{2}+\frac{\nu}{2}-\frac{\sigma}{2},\frac{\gamma}{2};1\right),\left(-\frac{\sigma}{2}+\frac{\delta}{2}+\frac{\nu}{2},\frac{\gamma}{2};1\right)\end{matrix}\right]. \quad (50)$$

Provided: $|\arg(z)| < \frac{1}{2}\pi\Omega, \Re(\sigma) > 0$ and $\Re(\delta) > 0$ is non negative integer.

7. Integral Involving Hypergeometric function

In the study of second order linear differential equation with three regular singular points, there arise the function

$$F\left(a,b;c;z\right)=1+\sum_{n=0}^{\infty}\frac{\left(a\right)_n\left(b\right)_n}{\left(c\right)_n}\frac{z^n}{n!}. \tag{51}$$

For c neither zero nor a negative integer in (51) the notation

$$\left(\alpha\right)_n=\alpha\left(\alpha+1\right)\left(\alpha+2\right)...\left(\alpha+n-1\right),n\geq 0.$$

is called the factorial function and the function in (51) is called the hypergeometric function ([9], p. 45).

$$I_{17}=\int_1^{\infty}x^{-\rho}\left(x-1\right)^{\sigma-1}{}_2F_1\left[\begin{array}{c}\nu+\sigma-\rho,\lambda+\sigma-\rho\\ \sigma\end{array};(1-x)\right]\bar{H}_{p,q}^{m,n}(zx)dx$$

$$=\frac{1}{2\pi i}\int_L\psi(x)z^s\{\int_1^{\infty}x^{-\rho+s}\left(x-1\right)^{\sigma-1}{}_2F_1\left[\begin{array}{c}\nu+\sigma-\rho,\lambda+\sigma-\rho\\ \sigma\end{array};(1-x)\right]dx\}ds.$$

Putting $x=t+1$ and $dx=dt$

$$=\sum_{k=0}^{\infty}\frac{\left(\nu+\sigma-\rho\right)_k\left(\lambda+\sigma-\rho\right)_k\left(-1\right)^k}{\left(\sigma\right)_k k!}$$

$$\times\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma\left(b_j-B_js\right)\prod_{j=1}^{n}\{\Gamma\left(1-a_j+A_js\right)\}^{\alpha_j}}{\prod_{j=m+1}^{q}\{\Gamma\left(1-b_j+B_js\right)\}^{\beta_j}\prod_{j=n+1}^{p}\Gamma\left(a_j-A_js\right)}$$

$$\times\frac{\Gamma\left(\sigma+k\right)\Gamma\left(\rho-s-k-\sigma\right)}{\Gamma\left(\rho-s\right)}z^s ds$$

$$=\Gamma\left(\sigma+k\right)\sum_{k=0}^{\infty}\frac{\left(\nu+\sigma-\rho\right)_k\left(\lambda+\sigma-\rho\right)_k\left(-1\right)^k}{\left(\sigma\right)_k k!}$$

$$\times\bar{H}_{p+1,q+1}^{m+1,n}\left[z\left|\begin{array}{c}\left(a_j,A_j;\alpha_j\right)_{1,n},\left(a_j,A_j\right)_{n+1,p},\left(\rho,1\right)\\ \left(b_j,B_j\right)_{1,m},\left(b_j,B_j;\beta_j\right)_{m+1,q},\left(\rho-k-\sigma,1\right)\end{array}\right.\right]. \tag{52}$$

Provided: $|\arg z|<\frac{1}{2}\pi\Omega$

Remark 1. *If we put $a_j=b_j=1,\forall i,j$ in various results then with the help of (5) it can be reduced in the form of the H-function and help of the equation (15) all finding results can be written in the form of $\bar{I}$ -functions.*

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