Acta Universitatis Apulensis No. 44/2015
ISSN: 1582-5329 pp. 167-173
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2015.44.11

APPLICATION OF SUPERORDINATION TO A SUBCLASS OF ANALYTIC FUNCTIONS INCLUDED DOUBLE INTEGRAL OPERATORS

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Abstract. We suppose that the normalized analytic function $f(z)$ satisfies the differential equation

$$f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z) = g(z),$$

where g is univalent in the open unit disk $\mathbb{D}$ and is superordinate to a convex-univalent function $h(z)$ normalized by $h(0) = 1$. In addition, we assume that the function $f(z)$ is given by a double integral operator of the form

$$f(z) = (1+\delta_1)(1+\delta_2)\int_0^1\int_0^1 s^{\delta_1} t^{\delta_2} z G'(zt^{\mu}s^{\nu})\, ds\, dt,$$

where $G'(z) + zG''(z) = g(z)$. We shall determine the best subordinant of the solutions of differential superordination

$$h(z) \prec f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z).$$

Some special cases are given in the corollaries.

2010 *Mathematics Subject Classification*: Primary 30C45, Secondary 30C80.

Keywords: convoulution, convex-univalent functions, integral operator, superordination.

1. Introduction

Let $\mathcal{A}$ be the class of all analytic functions $f(z)$ of the form

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots; \quad (z \in \mathbb{D}),$$

which satisfy the normalization condition $f(0) = f'(0) - 1 = 0$, and that $S \subseteq \mathcal{A}$ be the class of normalized univalent functions. Further, suppose that C denote the class of convex-univalent functions in $\mathbb{D}$. For two analytic functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \;\; g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

the Hadamard product (or convolution) of f and g is an analytic function in $\mathbb{D}$ defined by $(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$.

For $f, g \in \mathcal{A}$ the function f is subordinate to g (or g is superordinate to f) written as $f(z) \prec g(z)$ if there exist an analytic function $w(z)$ in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. If g is univalent in $\mathbb{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$, (see [3]).

Suppose that p, h are two analytic function in $\mathbb{D}$ and $\varphi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$. If $p(z)$ and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent in $\mathbb{D}$ and if $p(z)$ satisfies the second-order superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{1}$$

then p is called a solution of the differential superordination (1). An analytic function $q(z)$ is called a subordinant of (1), if $q(z) \prec p(z)$ for all the solutions of (1). The best subordinant $\tilde{q}$ is univalent subordinant that satisfies $q \prec \tilde{q}$ for all the subordinants q of (1), (see [4]).

Definition 1. ([3]) *We denote by Q the set of all functions $p(z)$ that are analytic and injective on $\overline{\mathbb{D}} \setminus E(p)$, where*

$$E(p) = \{\xi \in \partial\mathbb{D} : \lim_{z \to \xi} p(z) = \infty\},$$

and are such that $p'(\xi) \neq 0$ for $\xi \in \partial\mathbb{D} \setminus E(p)$.

We will use the following results, but we omit their proofs.

Lemma 1. ([5]) *Let $f, g \in \mathcal{A}$ and $F, G \in C$. If $f \prec F$ and $g \prec G$, then $f * g \prec F * G$.*

Lemma 2. ([4]) *Let $h(z)$ be convex in $\mathbb{D}$, with $h(0) = a, \lambda \neq 0$ and $\Re(\lambda) \geq 0$. If $p \in Q(a) = \{p \in Q : p(0) = a\}, p(z) + \frac{1}{\lambda} zp'(z)$ is univalent in $\mathbb{D}$ and*

$$h(z) \prec p(z) + \frac{1}{\lambda} zp'(z)$$

then $q(z) \prec p(z)$, where

$$q(z) = \frac{\lambda}{n z^{\lambda/n}} \int_0^z h(w) w^{\frac{\lambda}{n} - 1} dw.$$

The function q is convex in $\mathbb{D}$ and is the best subordinant.

In a recently paper [1] authors used subordination and investigated starlikeness and other properties of functions $f \in \mathcal{A}$ given by a double integral operator. In this article, using superordination, conditions on a different integral operator are investigated. Let $\delta_1 > -1$ and $\delta_2 > -1$. We consider functions $f \in \mathcal{A}$ defined by the double integral operator of the form

$$f(z) = (1+\delta_1)(1+\delta_2)\int_0^1\int_0^1 s^{\delta_1}t^{\delta_2} zG'(zt^{\mu}s^{\nu})\ ds\ dt;\ (G \in \mathcal{A}, z \in \mathbb{D}). \quad (2)$$

From (2) we see that

$$f'(z) = (1+\delta_1)(1+\delta_2)\int_0^1\int_0^1 s^{\delta_1}t^{\delta_2} g(zt^{\mu}s^{\nu})\ ds\ dt,$$

where $g(z) = G'(z) + zG''(z)$. In addition, we will see that there are suitable parameters α, λ such that

$$f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z) = g(z).$$

2. Main results

Let $h(z)$ be a convex-univalent function in $\mathbb{D}$ with $h(0) = 1$. For $\alpha \geq \lambda \geq 0$, consider $f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z)$ is univalent in $\mathbb{D}$. We define the class $S(\alpha, \lambda, h)$ of functions $f \in \mathcal{A}$ as following

$$S(\alpha,\lambda,h) = \{f \in \mathcal{A} : h(z) \prec f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z),\ z \in \mathbb{D}\}.$$

Put

$$\mu = \frac{1+\delta_2}{2}((\alpha-\lambda) - \sqrt{\triangle}),\quad \alpha - \lambda = \frac{\nu}{1+\delta_1} + \frac{\mu}{1+\delta_2},\quad (1+\delta_1)(1+\delta_2)\lambda = \mu\nu \quad (3)$$

where $\triangle = (\alpha-\lambda)^2 - 4\lambda$. It is seen that $\Re(\mu) \geq 0$ and $\Re(\nu) \geq 0$. Now we write the solution of

$$f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z) = g(z) \quad (4)$$

in it's double integral form. The relations (3) and (4) show that

$$\begin{aligned} g(z) &= f'(z) + \left(\frac{\mu\nu}{(1+\delta_1)(1+\delta_2)} + \frac{\nu}{1+\delta_1} + \frac{\mu}{1+\delta_2}\right) z f''(z) + \frac{\mu\nu}{(1+\delta_1)(1+\delta_2)} z^2 f'''(z) \\ &= \frac{\nu}{1+\delta_1} z^{1-\frac{1+\delta_1}{\nu}} \left(\frac{\mu}{1+\delta_2} z^{1+\frac{1+\delta_1}{\nu}} f''(z) + z^{\frac{1+\delta_1}{\nu}} f'(z)\right)' \\ &= \frac{\nu}{1+\delta_1} z^{1-\frac{1+\delta_1}{\nu}} \left(\frac{\mu}{1+\delta_2} z^{1+\frac{1+\delta_1}{\nu}-\frac{1+\delta_2}{\mu}} (z^{\frac{1+\delta_2}{\mu}} f'(z))'\right)'. \end{aligned}$$

Therefore

$$\frac{\mu}{1+\delta_2} z^{1+\frac{1+\delta_1}{\nu}-\frac{1+\delta_2}{\mu}} (z^{\frac{1+\delta_2}{\mu}} f'(z))' = \frac{1+\delta_1}{\nu} \int_0^z w^{\frac{1+\delta_1}{\nu}-1} g(w)\, dw.$$

Using the change of variable $w = zs^{\nu}$, we obtain

$$(z^{\frac{1+\delta_2}{\mu}} f'(z))' = \frac{(1+\delta_1)(1+\delta_2)}{\mu} \int_0^1 s^{\delta_1} z^{\frac{1+\delta_2}{\mu}-1} g(zs^{\nu})\, ds.$$

Integrating both sides, a change of variable yields

$$f'(z) = (1+\delta_1)(1+\delta_2) \int_0^1 \int_0^1 s^{\delta_1} t^{\delta_2} g(zt^{\mu} s^{\nu})\, ds\, dt.$$

Take $\psi_{\delta,\lambda}(z) = \int_0^1 \frac{t^{\delta}\, dt}{1-zt^{\lambda}}$. By Theorem [[3], 2.6h] it is seen that $\psi_{\delta,\lambda}(z) \in C$ provided that $\Re(\lambda) \geq 0$.

Theorem 3. Let μ and ν be defined as (3) and

$$q(z) = (1+\delta_1)(1+\delta_2) \int_0^1 \int_0^1 s^{\delta_1} t^{\delta_2} h(zt^{\mu} s^{\nu})\, ds\, dt. \tag{5}$$

Then the function $q(z) = (1+\delta_1)(1+\delta_2)(\psi_{\delta_1,\nu} * \psi_{\delta_2,\mu} * h)(z)$ is convex. If $f \in S(\alpha, \lambda, h)$, $f'(z) \in Q$ and $f'(z) + \frac{\nu}{1+\delta_1} z f''(z) \in Q$ then $q(z) \prec f'(z)$ and q is the best subordinant.

Proof. We have

$$\psi_{\delta_2,\mu}(z) * h(z) = \int_0^1 \frac{t^{\delta_2}\, dt}{1-zt^{\mu}} * h(z) = \int_0^1 t^{\delta_2} h(zt^{\mu})\, dt.$$

Therefore

$$\begin{aligned} (\psi_{\delta_1,\nu}(z) * \psi_{\delta_2,\mu}(z)) * h(z) &= \psi_{\delta_1,\nu}(z) * \int_0^1 t^{\delta_2} h(zt^{\mu})\, dt \\ &= \int_0^1 s^{\delta_1} \left(\int_0^1 t^{\delta_2} h(zs^{\nu} t^{\mu})\, dt \right) ds \\ &= \int_0^1 \int_0^1 s^{\delta_1} t^{\delta_2} h(zt^{\mu} s^{\nu})\, ds)\, dt. \end{aligned}$$

The function $q(z)$ is convex, since the functions $\psi_{\delta_1,\nu}, \psi_{\delta_2,\mu}$ and h are convex univalent in $\mathbb{D}$ (see [2]). Put $p(z) = f'(z) + \frac{\nu}{1+\delta_1} z f''(z)$, then $h(z) \prec p(z) + \frac{\mu}{1+\delta_2} z p'(z)$. By Lemma 2 we obtain

$$\frac{1+\delta_2}{\mu z^{\frac{1+\delta_2}{\mu}}} \int_0^z w^{\frac{1+\delta_2}{\mu}-1} h(w)\, dw = (1+\delta_2)(\psi_{\delta_2,\mu}(z) * h(z)) \prec p(z),$$

or equivalently

$$(1+\delta_2)(\psi_{\delta_2,\mu}(z) * h(z)) \prec f'(z) + \frac{\nu}{1+\delta_1} z f''(z).$$

Using again Lemma 2 we obtain

$$\frac{1+\delta_1}{\nu z^{\frac{1+\delta_1}{\nu}}} \int_0^z (1+\delta_2) w^{\frac{1+\delta_1}{\nu}-1} (\psi_{\delta_2,\mu} * h)(w)\, dw \prec f'(z)$$

or equivalently $q(z) \prec f'(z)$. Since $q(z) + \alpha z q'(z) + \lambda z^2 q''(z) = h(z)$, this means that $q(z)$ is a solution of the differential superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z) = p(z) + \alpha z p'(z) + \lambda z^2 p''(z) \tag{6}$$

which $f'(z)$ also satisfies (6). Therefore $q(z)$ will be a dominant for all subordinants of $h(z) \prec f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z)$. Hence $q(z)$ is the best subordinant of it.

Corollary 4. Suppose that all conditions of Theorem 3 are satisfied. Then

$$(1+\delta_1)(1+\delta_2) \int_0^1 \int_0^1 \int_0^1 s^{\delta_1} t^{\delta_2} h(zrt^{\mu} s^{\nu})\, dr\, dt\, ds = \int_0^1 q(tz) dt \prec \frac{f(z)}{z}.$$

Proof. Consider $p(z) = \frac{f(z)}{z}$, then $q(z) \prec p(z) + zp'(z) = f'(z)$. Lemma 2 shows that

$$\int_0^1 q(tz) dt = \frac{1}{z} \int_0^z q(w) dw \prec p(z) = \frac{f(z)}{z}.$$

Using Theorem 3 and Corollary 4 with $h(z) = \frac{1+Az}{1+Bz}$ where $-1 \le B < A \le 1$, we obtain the following result.

Corollary 5. Suppose that all conditions of Theorem 3 are satisfied. If

$$\frac{1+Az}{1+Bz} \prec f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z)$$

then $q(z; A, B) \prec f'(z)$, where

$$q(z; A, B) = \frac{A}{B} - \frac{(1+\delta_1)(1+\delta_2)(A-B)}{B} \int_0^1 \int_0^1 \frac{s^{\delta_1} t^{\delta_2}\, ds\, dt}{1 + Bzt^{\mu} s^{\nu}};\ (B \neq 0)$$

and

$$q(z; A, 0) = 1 + \frac{A(1+\delta_1)(1+\delta_2) z}{(1+\delta_1+\nu)(1+\delta_2+\mu)} \prec f'(z),$$

also the functions $q(z; A, B)$ and $q(z; A, 0)$ are the best subordinants. In addition

$$\frac{A}{B} - \frac{(1+\delta_1)(1+\delta_2)(A-B)}{B} \int_0^1 \int_0^1 \int_0^1 \frac{s^{\delta_1} t^{\delta_2}\, dr\, ds\, dt}{1 + Bzrt^{\mu}s^{\nu}} \prec \frac{f(z)}{z}$$

if $B \neq 0$, and

$$1 + \frac{A(1+\delta_1)(1+\delta_2)z}{2(1+\delta_1+\nu)(1+\delta_2+\mu)} \prec \frac{f(z)}{z}$$

for $B = 0$.

Finally, the last theorem is about the convolution of two functions in $S(\alpha, \lambda, h)$.

Theorem 6. Let μ and ν are given by (3) and $f, g \in S(\alpha, \lambda, h)$. If $g'(z) \in Q$, $g'(z) + \frac{\nu}{1+\delta_1} z g''(z) \in Q$ and $f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z)$, $\frac{g(z)}{z} \in C$, then $f * g$ belongs to $S(\alpha, \lambda, h_1)$ where $h_1(z) = q(z) * \int_0^1 h(tz)dt$ and $q(z)$ is given by (5).

Proof. It is easy to see that

$$(f*g)'(z) + \alpha z (f*g)''(z) + \lambda z^2 (f*g)'''(z) = (f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z)) * \frac{g(z)}{z}.$$

Hence

$$\begin{aligned} h_1(z) &= q(z) * \int_0^1 h(tz)\, dt \\ &= (1+\delta_1)(1+\delta_2)(h(z) * \psi_{\delta_1,\nu}(z) * \psi_{\delta_2,\mu}(z)) * (h(z) * \psi_1(z)) \\ &= (1+\delta_1)(1+\delta_2)(h(z) * \int_0^1 \int_0^1 \int_0^1 s^{\delta_1} t^{\delta_2} h(zrt^{\mu}s^{\nu})\, dr\, ds\, dt) \\ \text{(by Lemma 1)} &\prec (f'(z) + \alpha z f''(z) + \lambda z^2 f'''(z)) * \frac{g(z)}{z} \\ &= (f*g)'(z) + \alpha z (f*g)''(z) + \lambda z^2 (f*g)'''(z), \end{aligned}$$

where $\psi_1(z) = \int_0^1 \frac{dr}{1-zr}$. This completes the proof.

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