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APPLICATIONS OF CARLSON SHAFFER OPERATOR IN UNIVALENT FUNCTION THEORY

K. I. Noor and S. Riaz

Abstract. In this research paper, we introduce some new classes of k-starlike functions and k-uniformly close-to-convex functions in the unit disk $E = \{z : |z| < 1\}$ by using Carlson-Sheffer operator. Some inclusion relationships, coefficient bounds and other interesting properties of these classes are investigated. Some known results are derived as special cases.

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1. Introduction

Let A be the class of functions $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad |z| < 1. \tag{1.1}$$

analytic in $E = \{z : |z| < 1\}$. Let S, C , S^* , K be the subclasses of A of univalent, convex, starlike and close-to-convex functions respectively. The convolution (Hadamard product) given by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad |z| < 1, \tag{1.2}$$

where $f(z)$ is given by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, see [2].

Let f and g be analytic in E. The function f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if g is univalent in E, $f(0) = g(0)$ and $f(E) \subset g(E)$, see [7].

Let incomplete beta function $\phi(a,c;z)$, see [9] defined by

$$\phi(a,c;z) = z\,{}_2F_1(1,a,c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}} z^n, \quad |z| < 1,\ c \neq 0,-1,-2...., \tag{1.3}$$

where $(a)_n$ is Pochhammer symbol defined interms of the Gamma functions, by

$$(\alpha)_k = \frac{\Gamma(a+n)}{\Gamma(n)} = \begin{cases} 1,\ n=0, \\ n(n+1)(n+2)....(a+n-1),\ n \in N. \end{cases} \tag{1.4}$$

Further for $f(z) \in A$, then a linear operator $L(a,c) : A \to A$, see [1] defined as

$$L(a,c)f(z) = \phi(a,c;z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}} z^n, \quad |z| < 1, \tag{1.5}$$

where $\phi(a,c;z)$ is given by (1.3). It follows from (1.3) and (1.5) that

$$z(L(a,c)f(z))' = aL(a+1,c)f(z) - (a-1)L(a,c)f(z). \tag{1.6}$$

$L(a,c)f$ is a polynomial for $a = 0,-1,-2,....$. For $a \neq 0.-1,-2,...,$ root test implies that

$$\lim_{n\to\infty} \left| \frac{(a)_n}{(c)_n} \right|^{\frac{1}{n}} = 1.$$

This shows that infinite series for $L(a,c)f$ and f has same radius of convergence. There is $1-1$ mapping of A onto itself with $L(a,a)$ as identity and $L(c,a)$ is the continuous inverse of $L(a,c)$ $(a \neq 0.-1,-2,...)$. Furthermore, if $h(z) = zf'(z)$, then $f(z) = L(1,2)h(z)$ and $h(z) = L(2,1)f(z)$. Carlson-Shaffer operator generalizes other linear operators.

In 1999, Kanas and Wisniowska [3] introduced the conic domain Ω_k, $k \geq 0$ and studied it comprehensively, defined as

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}. \tag{1.7}$$

Extremal functions for the conic regions Ω_k are given as

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, \quad k=0, \\ 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \quad k=1, \\ 1 + \frac{2}{1-k^2}\sinh^2[(\frac{2}{\pi}\arccos k)\operatorname{arctanh}\sqrt{z}], \quad 0<k<1, \\ 1 + \frac{1}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right) + \frac{1}{k^2-1},\ k>1, \end{cases} \tag{1.8}$$

where, $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0,1), |z| < 1$ and z can be chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is Legendre's complete elliptic integral of $R(t)$, see [3], [4].

If $p_k(z) = 1 + \delta_k z + ...$, then from (1.8) one can have

$$\delta_k = \begin{cases} \frac{8(\arccos k)^2}{\Pi^2(1-k^2)}, & 0 \le k < 1 \\ \frac{8}{\Pi^2}, & k = 1 \\ \frac{\pi^2}{4(k^2-1)\sqrt{t(1+t)k^2(t)}}, & k > 1 \end{cases} \tag{1.9}$$

Later, Kanas and Wisniowska [3] defined the class of functions which maps open unit disk $|z| < 1$ into these conic regions and denoted this class by $P(p_k)$ as, p satisfy the condition $p(0) = 1$ belongs to the class $P(p_k)$, if $p(z) \prec p_k(z)$, $|z| < 1$. That is, $p(E) \subset p_k(E) = \Omega_k$. $p(z) \in P(p_k)$ holds following property that $\Re e\,(p(z)) > \frac{k}{k+1}$.

Now we define the following classes.

Definition 1.1 If a function f is analytic in $|z| < 1$ and defined by *(1.1)*, then $f \in k - UCV(a,c)$ if and only if

$$L(a,c)f \in k - UCV \quad (c \neq 0, -1, -2, ...). \tag{1.10}$$

Special Cases

(i) $0 - UCV(1,1) \equiv C$, see [17].

(ii) $k - UCV(1,1) \equiv k - UCV$, we refer [3].

Definition 1.2 If f is analytic in $|z| < 1$ and defined by *(1.1)*, then $f \in k - UT(a,c)$ if and only if

$$L(a,c)f \in k - ST \quad (c \neq 0, -1, -2, ...). \tag{1.11}$$

Special Cases

(i) $0 - UT(a,c) \equiv T(a,c)$ introduced and studied in [18],

(ii) $0 - UT(1,1) \equiv S^*$, see [12].

(iii) $k - UT(2,1) \equiv k - UCV$, we refer [3].

(iv) $k - UT(1,1) \equiv k - ST$ introduced and studied in [3].

(v) $0 - UT(2,1) \equiv C$, see [17].

The relationship between the classes of is $k - UCV(a,c)$ and $k - UT(a,c)$ is given as

$$f \in k - UCV(a,c) \quad if\ and\ only\ if \quad zf' \in k - UT(a,c). \tag{1.11}$$

Definition 1.3 If f is analytic in $|z| < 1$ and defined by *(1.1)* , then $f \in k - UK(a, c)$ if and only if

$$L(a, c)f \in k - UK \qquad (c \neq 0, -1, -2, ...). \tag{1.12}$$

Special Cases

(i) $0 - UK(1, 1) \equiv K$,
(ii) $k - UK(1, 1) \equiv k - UK$, see [14].
(iii) $0 - UK(2, 1) \equiv C^*$, we refer [15].
(iv) $k - UK(1, 1) \equiv k - UC^*$, introduced in [14].
(v) We take $g(z) = f(z)$ in (1.12), we obatin the class $k - UCV(a, c)$.

Definition 1.4 If f is an analytic function in $|z| < 1$ and defined by *(1.1)* , then $f \in k - UC^*(a, c)$ if and only if

$$L(a, c)f \in C^* \quad (c \neq 0, -1, -2, ...). \tag{1.13}$$

Special Cases

(i) $0 - UC^*(1, 1) \equiv C^*$, see [15].
(ii) $k - UC^*(1, 1) \equiv k - UC^*$, we refer [14].
(iii) We take $g(z) = f(z)$ in (1.13), we obatin the class $k - UCV(a, c)$.

The relationship between the classes of is $k - UC^*(a, c)$ and $k - UK(a, c)$ is given as

$$f \in k - UC^*(a, c) \quad if\ and\ only\ if \quad zf' \in k - UK(a, c). \tag{1.14}$$

2. Preliminary Concepts

To prove our results, we need the following lemmas.

Lemma 2.1 [5] Let $f(z) = z + a_2z^2 + a_3z^3 + ... \in k - ST$. Then

$$|a_2| \leq |\delta_k|.$$

This coefficient bound is also holds for the classes of $k - UCV$, $k - UK$ and $k - UC^*$.

Lemma 2.2 [11] If a , b and c are real and satisfy

$$-1 \leq a \leq 1, b \geq 0 \quad \text{and} \quad c > 1 + \max\{2 + |a + b - 2|, 1 - (a - 1)(b - 1)\},$$

then

$$zF(a,b;c;z) \in S^*, \tag{2.1}$$

where

$$F(a,b;c;z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

is the Guassian hypergeometric function.

Lemma 2.3 [18] If a and c are real and satisfy

$$-1 \leq a \leq 1 \quad and \quad c > 3 + |a|,$$

then $\phi(a,c;z)$ defined by (1.3) is convex in E.

Lemma 2.4 [16] The class S^* and K are closed under convex convolution.

Lemma 2.5 [5] Let $0 \leq k < \infty$ and β, δ be any complex numbers with $\beta \neq 0$ and $\Re(\frac{\beta k}{k+1} + \delta) > \delta$ where γ is defined as:
If $h(z)$ is analytic in E, $h(0) = 1$ and it satisfies

$$h(z) + \frac{zh'(z)}{\beta h(z) + \delta} \prec p_{k,\gamma}, \tag{2.2}$$

and $q_{k,\gamma}$ is an anlytic solution of

$$q_{k,\gamma}(z) + \frac{zq'_{k,\gamma}(z)}{\beta q_{k,r}(z) + \delta} = p_{k,r}(z), \tag{2.3}$$

then $q_{k,\gamma}(z)$ is univalent, $h(z) \prec q_{k,\gamma}(z) \prec p_{k,\gamma}(z)$, and $q_{k,\gamma}(z)$ is best dominant of (2.2).

Lemma 2.6 [17] If $f(z) \in C$ and $g \in S^*$, then sor any analytic function in E with $F(0) = 1$.

$$\frac{f * Fg}{f * g}(E) \subset \overline{C_0}F(E), \quad f \in C, \ g \in S^*, \tag{2.4}$$

where $\overline{C_0}F(E)$ denotes the closed convex hull of $F(E)$ (the smalest convex set which contain $F(E)$).

Lemma 2.7 [10] Let P be a complex function in E, with $\Re e(P(z)) > 0$ for $z \in E$ and h be a convex function in E. If $p(z)$ be a analytic function in E, with $p(0) = h(0)$ then,

$$p(z) + P(z)zp'(z) \prec h(z). \tag{2.5}$$

3. Main Results

Theorem 3.1 For $a \geq 1$

$$k - UT(a+1, c) \subset k - UT(a, c).$$

Proof. Let $f(z) \in k - UT(a+1, c)$.
Let

$$\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} = p(z). \tag{3.1}$$

Then $p(z)$ is analytic with $p(0) = 1$.
From (2.6) and (3.1), we have

$$aL(a+1,c)f(z) - (a-1)L(a,c)f(z) = p(z)L(a,c)f(z),$$

or

$$aL(a+1,c)f(z) = L(a,c)f(z)[(a-1) + p(z)].$$

Differentiating logarithmically, we get

$$\begin{aligned} \frac{a(L(a+1,c)f(z))'}{L(a+1,c)f(z)} &= \frac{z(L(a,c)f(z))'}{L(a,c)f(z)} + \frac{zp'(z)}{p(z)+(a-1)} \\ &= p(z) + \frac{zp'(z)}{p(z)+(a-1)}. \end{aligned} \tag{1}$$

Since $f \in k - UT(a+1, c)$, it follows that

$$\left\{ p(z) + \frac{p'(z)}{p(z)+(a-1)} \right\} \prec p_k(z),$$

and by using Lemma , $p(z) \prec p_k(z)$. This proves that $f(z) \in k - UT(a, c)$ in E.

As special case we note that for $k = 0$ in Theorem 3.1, we obtain the known result given in [18].

Theorem 3.2 Let $f(z) \in k - UT(a, c)$ and

$$F(z) = \frac{\gamma+1}{z^{\gamma}} \int_0^z t^{\gamma-1} f(t)dt \quad (\gamma \geq 0). \tag{3.3}$$

Then $F(z) \in k - UT(a, c)$.

Proof. From (3.3), we note that $F(z) \in A$ and

$$r(L(a,c)F(z)) + z(L(a,c)F(z))' = (\gamma+1)L(a,c)f(z). \tag{3.4}$$

Let

$$h(z)=\frac{z(L(a,c)F(z))'}{L(a,c)F(z)}. \tag{3.5}$$

We note that $h(z)$ is analytic in E write $h(0)=1$.
Then, from (3.4), we have

$$r+h(z)=(r+1)\frac{L(a,c)f(z)}{L(a,c)F(z)}.$$

Differentiating Logarithimacally, we get

$$h(z)\prec p_k(z) \text{ in } E,$$

and this proves that $F(z)\in k-UT(a,c)$ in E.

Theorem 3.3 For $a\geq 1$,

$$k-UK(a+1,c)\subset k-UK(a,c).$$

Proof. Let $f(z)\in k-UK(a+1,c)$. Then there exists $g(z)\in k-UT(a+1,c)$ such that

$$\frac{z(L(a+1,c)f(z))'}{L(a,c)g(z)}=p(z). \tag{3.6}$$

Using (1.6), we have

$$aL(a+1,c)f(z)-(a-1)L(a,c)f(z)=p(z)(L(a,c)g(z)),$$

and differentiating we get

$$\begin{aligned} a(L(a+1,c)f(z))' &= p'(z)(L(a,c)g(z))+(a-1)(L(a,c)f(z))'+p(z)(L(a,c)g(z))' \\ &= p'(z)(L(a,c)g(z))+(a-1)(L(a,c)f(z))' \\ &\quad +p(z)[aL(a+1,c)g(z)-(a-1)L(a,c)g(z). \end{aligned} \tag{2}$$

Using (1.6), we can write

$$\begin{aligned} \frac{z(L(a+1,c)f(z))'}{L(a+1,c)g(z)} &= zp'(z)\left\{\frac{L(a,c)g(z)}{aL(a+1,c)g(z)}\right\} \\ &\quad +(a-1)\left\{\frac{(L(a,c)f(z))'}{L(a,c)g(z)}.\frac{L(a,c)g(z)}{L(a+1,c)g(z)}\right\} \\ &\quad +p(z)\left\{1-(a-1)\frac{(L(a,c)g(z))'}{L(a,c)g(z)}.\frac{L(a,c)g(z)}{L(a+1,c)g(z)}\right\}. \end{aligned} \tag{3}$$

Since $g(z) \in k-UT(a+1,c)$ and $k-UT(a+1,c) \subset k-UT(a,c)$, it follows that

$$\frac{z(L(a,c)g(z))'}{L(a,c)g(z)} = p_0(z) \prec p_k(z).$$

From (**??**), (**??**), we get

$$\frac{z(L(a+1,c)f(z))'}{L(a+1,c)g(z)} = p(z) + \frac{zp'(z)}{p_0(z)+(a-1)}. \tag{3.7}$$

Now $p_0(z) \in P(p_k) \subset P\left(\frac{k}{k+1}\right) \subset P$ and $a \geq 1$, so $\Re(p_0(z)+(a-1)) > 0$. Let $h_0(z) = \frac{1}{p_0(z)+(a-1)}$. Then $\Re h_0(z) > 0$ in E.
Thus, from (3.9) and $f(z) \in k-UK(a+1,c)$, we obtain

$$[p(z) + h_0(z)(zp'(z))] \prec p_k(z).$$

Using Lemma 2.7, it gives us that

$$p(z) \prec p_k(z),$$

which proves that $f(z) \in k-UK(a,c)$ in E. This completes the proof.

Theorem 3.4 For $F(z)$ be defined by (3.3) and $f(z) \in k-UK(a,c)$, $z \in E$. Then

$$F(z) \in k-UK(a,c).$$

Proof. Since $f(z) \in k-UK(a,c)$, there exists $g(z) \in k-UT(a,c)$ such that $\frac{z(L(a,c)f(z))'}{L(a,c)g(z)} \prec p_k(z), z \in E$.
Let

$$G(z) = \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1}g(t)dt \qquad (\gamma \geq 0). \tag{3.10}$$

Then, by Theorem 3.2 leads us that $G(z) \in k-UT(a,c)$ in E.
Let

$$H(z) = \frac{z(L(a,c)F(z))'}{L(a,c)G(z)}. \tag{3.11}$$

Then $H(z)$ is analytic in E with $H(0) = 1$.
From (3.10) and (3.11), we have

$$H'(z)(L(a,c)G(z)) + H(z)(L(a,c)G(z))' = -\gamma(L(a,c)F(z))' + (\gamma+1)(L(a,c)f(z))'.$$

This gives us

$$zH'(z)+H(z)\frac{z(L(a,c)G(z))'}{L(a,c)G(z)}=-\gamma\frac{z(L(a,c)F(z))'}{L(a,c)G(z)}+(\gamma+1)\frac{\frac{z(L(a,c)f(z))'}{L(a,c)g(z)}}{\frac{L(a,c)G(z)}{L(a,c)g(z)}}. \quad (3.12)$$

Let $\frac{(L(a,c)G(z))'}{L(a,c)G(z)}=p_0(z), p_0(z)\in P(p_k)\subset P$ and so $\Re e(p_0(z)+\gamma)\in P$ in E. It follows that

$$\left\{H(z)+\frac{zH'(z)}{p_0(z)+\gamma}\right\}\prec p_k(z).$$

From this we have

$$H(z)+h_1(z)(zH'(z))\prec p_k(z),$$

where $h_1(z)=\frac{1}{p_0(z)+\gamma}\in P$.

We now apply Lemma 2.7, and this gives us $H(z)\prec p_k(z)$, which proves that $F(z)\in k-UK(a,c)$ in E.

Theorem 3.5 Let $f\in k-UT(a,c)$ and $\phi\in C$, then $\phi * f\in k-UT(a,c)$.

Proof. Let

$$\frac{z\left[L(a,c)(f*\phi)(z)\right]'}{L(a,c)(f*\phi)(z)}=\frac{z(L(a,c)f(z))'*\phi(z)}{(L(a,c)f(z)*\phi(z)}$$

$$=\frac{\phi(z)*\frac{z(L(a,c)f(z))'}{L(a,c)f(z)}L(a,c)f(z)}{\phi(z)*L(a,c)f(z)}$$

$$=\frac{\phi(z)*h(z)(L(a,c)f(z))}{\phi(z)*L(a,c)f(z)}.$$

Now $\phi\in C$, $L(a,c)f(z)\in k-UT \subset S^*$, $h(z)\in P(p_k)$, so using Lemma 2.6 we have

$$\frac{z(L(a,c)(f*\phi)'}{L(a,c)(f*\phi)}\in P(p_k),$$

and therefore $\phi * f\in k-UT(a,c)$.

Special Cases

(i) We take $k=0$, it follows that $S(a,c)$ is invariant under convex convolution.

(ii) For $a=1$, $c=1$ and $k=0$, we get the well known result that the class S^* is closed under convolution with convex function. For this we refer [17].

Following the similar techniques, we can easily prove the following.

Theorem 3.6 Let $\phi\in C$ and let $f\in k-UK(a,c)$. Then $\phi * f\in k-UK(a,c)$. (We include the proof for the sake of completeness).

Proof. Since $f\in k-UK(a,c)$, $\frac{z(L(a,c)f(z))'}{L(a,c)g(z)}\in P(p_k)$, $g\in k-UT(a,c)$.

$$\begin{aligned} \frac{z\,[L(a,c)(f*\phi)(z)]'}{L(a,c)(g*\phi)(z)} &= \frac{\phi(z)*\frac{z(L(a,c)f(z))'}{L(a,c)g(z)}L(a,c)g(z)}{\phi(z)*L(a,c)g(z)} \\ &= \frac{\phi(z)*h(z)(L(a,c)g(z))}{\phi(z)*L(a,c)g(z)}, \end{aligned}$$

where $\phi \in C$, $h \in P(p_k)$, $L(a,c)g \in S^*$. Now on using Lemma 2.6, we obtain the required result that $(f*\phi) \in k-UK(a,c)$ in E.

As special cases we note that when $a=1$, $c=1$ and $k=0$ in Theorem 3.6, it follows that the class K is closed under convolution with convex function, see [17].

Applications of Theorem 3.5 and Theorem 3.6

From Theorem 3.5 and Theorem 3.6, it follows that the classes $k-UT(a,c)$ and $k-UK(a,c)$ are invariant under convolution with convex function. Using this fact, it can be easily verified that these classes are closed under the integral operators given as:

(i) $f_1(t) = \int_0^z \frac{f(t)}{t}dt.$

(ii) $f_2(t) = \frac{2}{z}\int_0^z f(t)dt.$

(iii) $f_3(t) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt.$

As applications of Theorem 3.5 and Theorem 3.6 we have following results.

Theorem 3.7 Let a and c be real and satisfy

$$c \neq 0, -1 < c \leq 1, \text{and } a > 3+|c|. \tag{3.13}$$

Then

$$k-UT(a,c) \subset k-ST.$$

Proof. If $f(z) \in k-UT(a,c)$. That is $L(a,c)f(z) = \phi(a,c)*f(z) \in k-ST$. Since a and c satisfy the condition (3.13), we have from that $\phi(c,a) \in C$. Therefore, an application of Theorem 3.5 leads to

$$f = \phi(c,z)*\phi(a,c)f \in k-ST.$$

As special case we take $k=0$, then we obtain the known result given in [18].

Using Theorem 3.6 and similar techniques we have the following.

Theorem 3.8 Let a and c be real and satisfy (3.13). Then

$$k-UK(a,c)\subset k-UK.$$

Theorem 3.9 Let a, c and d be real. If

$$d\neq 0, -1<d\leq 1 \text{ and } c>3+|d|, \tag{3.14}$$

then
(*i*) $k-UT(a,d)\subset k-UT(a,c)$,
(*ii*) $k-UK(a,d)\subset k-UK(a,c)$.
Proof. Let

$$f(z)\in k-UT(a,d).$$

Then

$$L(a,d)f(z)=\phi(a,d)*f(z)\in k-ST.$$

Using Lemma 2.3, $\phi(d,c)\in C$. Hence,

$$\begin{aligned} L(a,c)f(z) &= \phi(a,c)*f(z)\\ &= \phi(a,d)*\phi(d,c)*f(z)\\ &= \phi(d,c)*\phi(a,d)*f(z). \end{aligned}$$

Since $\phi(a,d)*f(z)=L(a,d)f(z)\in k-ST$ and $\phi(d,c)\in C$, it follows $L(a,c)f(z)\in k-ST$ and consequently $f(z)\in k-UT(a,c)$. This completes the proof.
Proof of (ii) is similar and therefore omit it.

As special case we take $k=0$ in Theorem 3.9, this implies the following.
(*i*) $S(a,d)\subset S(a,c)$ which has been proved in [18].
(*ii*) $K(a,d)\subset K(a,c)$.

Theorem 3.10 Let $f\in k-UT(a,c)$ and $f(z)$ be given by (1.1). Then

$$|a_2|\leq \left|\frac{c}{a}\right|\delta_k,$$

Proof. Since we have $L(a,c)f(z)=z+\sum\limits_{n=1}^{\infty}\frac{(a)_n}{(c)_n}a_{n+1}z^{n+1}$ belongs to $k-ST$, this implies that

$$|\frac{aa_2}{c}|\leq \delta_k,$$

which gives the required result.

Special Cases

(i) We take $k=0$, we have $\delta_k=2$. This implies that $|a_2|\leq 2|\frac{c}{a}|$.

(ii) For $k=1$, we have $\delta_k=\frac{8}{\pi^2}$, from which follows that $|a_2|\leq|\frac{c}{a}|\frac{8}{\pi^2}$.

(iii) We take $a=2$ and $c=1$, it follows that $L(2,1)f=zf'$. Therefore, we have $L(2,1)f\in k-ST$ implies that $|a_2|\leq\frac{\delta_k}{4}$.

Let a and c satisfy condition (3.13). Then by Theorem 3.7, $f\in k-UT(a,c)$ is starlike and hence univalent. Using this observation, we prove the following covering result.

Theorem 3.11 Let a and c satisfy (3.13) and let $f\in k-UT(a,c)$. Then $f(E)$ contains the disk

$$|w|<\frac{a}{2a+|c|\delta_k}. \tag{3.15}$$

Proof. Since $f\in k-UT(a,c)$ with a and c defined by (3.13) is univalent,

$$g(z)=\frac{w_0f(z)}{w_0-f(z)}=z+\left(a_2+\frac{1}{w_0}\right)z^2+(a_3+\frac{1}{w_0{}^2})z^3+.....$$

is also univalent, where w_0 $(w_0\neq 0)$ is complex number such that $f(z)\neq w_0$ for $z\in E$. Hence

$$\left|\frac{1}{|w_0|}-|a_2|\right|\leq\left|a_2+\frac{1}{w_0}\right|\leq 2.$$

Now, using Theorem 3.10, we have $|a_2|\leq|\frac{c}{a}|\delta_k$, where δ_k is given by (1.9). This gives us

$$\frac{1}{|w_0|}\leq 2+\left|\frac{c}{a}\right|\delta_k=\frac{2a+|c|\delta_k}{a},$$

which implies that

$$|w_0|\geq\frac{2a+|c|\delta_k}{a}.$$

This completes the proof of theorem.

Special Cases

(i) We take $k=0$, we have $\delta_k=2$. It follows that, $f(E)$ contains the disk $|w|\leq\frac{a}{2(a+|c|)}$ which has been proved in [18].

(ii) For $k=1$, we have $\delta_k=\frac{8}{\pi^2}$. That is $f\in 1-UT(a,c)$ implies that $f(E)$ contains the disk $|w|\leq\frac{a\pi^2}{2(a\pi^2+4)}$.

(iii) We take $a = 2$ and $c = 1$, it follows that $L(2,1)f = zf'$. Therefore, we have $L(2,1)f \in k - ST$ implies that $f(E)$ contains the disk $|w| < \frac{4}{8+\delta_k}$.

Theorem 3.12 Let $f \in k - UT(a,c)$ and for $\alpha \geq 0$, let

$$F_\alpha(z) = (1-\alpha)f(z) + \alpha z f'(z).$$

Then $F_\alpha(z) \in k - UT(a,c)$ for $|z| < r_\alpha$, where

$$r_\alpha = \frac{1}{2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1}}. \tag{3.16}$$

Proof. When $\alpha = 0$, the proof is immediate. So we take $\alpha > 0$. In Theorem 3.5, we have proved that the class $k - UT(a,c)$ is preserved under convex convolution. We define

$$\begin{aligned} \phi_\alpha(z) &= (1-\alpha)\frac{z}{(1-z)} + \alpha\frac{z}{(1-z)^2} \\ &= z + \sum_{n=2}^{\infty}(1+(n-1)\alpha)z^n. \end{aligned} \tag{4}$$

It is known [10] and can easily be verified that $\phi_\alpha(z) \in C$ for $|z| < r_\alpha$, where r_α is given by (3.16).
We can write

$$F_\alpha(z) = (1-\alpha)f(z) + \alpha z f'(z) = \phi_\alpha(z) * f(z).$$

Since $f \in k - UT(a,c)$, $\phi_\alpha \in C$ in $|z| < r_\alpha$, therefore , by Theorem 3.5, it follows that $F_\alpha \in k - UT(a,c)$ in $|z| < r_\alpha = \frac{1}{2\alpha+\sqrt{4\alpha^2-2\alpha+1}}$.

Special Cases

(*i*) Let $\alpha = \frac{1}{2}$ in Theorem 4.2.8. Then we have $F_\alpha(z) = \frac{(zf(z))'}{2}$. This is Livingston's operator, see [8]. In this case, $r_{\frac{1}{2}} = \frac{1}{2}$.

(*ii*) For $\alpha = 1$ in Theorem 4.2.6. It follows that $F_\alpha(z) = zf'(z)$ and $f \in k - UT(a,c)$. In this case $F_\alpha(z) \in k - UT(a,c)$ for $|z| < r_1 = \frac{1}{2+\sqrt{3}}$.

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Khalida Inayat Noor
Department of Mathematics, COMSATS Institute of Information Technology,
Islamabad, Pakistan
email: *khalidanoor@gmail.com*

Sadia Riaz
Department of Mathematics, COMSATS Institute of Information Technology,
Islamabad, Pakistan
email: *sadia_riaz2007@yahoo.com*