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ON A CERTAIN SUBCLASS OF MEROMORPHIC FUNCTIONS DEFINED BY HILBERT SPACE OPERATOR

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Abstract. In this work, using Hilbert space operator we define a new subclass of meromorphic functions and determine coefficient estimates, radii of starlikeness, and convexity for the functions in this class.

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1. Introduction

Let Σ denote the class of functions

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1}$$

which are analytic in the punctured unit disk

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\},$$

with a simple pole at the origin.

For functions $f \in \Sigma$ given by (1) and $g \in \Sigma$ defined by

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n, \tag{2}$$

the Hadamard product (or convolution) [2] of f and g is given by

$$(f * g)(z) := \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n =: (g * f)(z).$$

Lashin [4] defined the following integral operator $Q_\beta^\gamma : \Sigma \to \Sigma$:

$$Q_\beta^\gamma = Q_\beta^\gamma f(z) = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)} \frac{1}{z^{\beta+1}} \int_0^z t^\beta \left(1-\frac{t}{z}\right)^{\gamma-1} f(t)\,dt \quad (\beta>0, \gamma>1;\ z\in \mathbb{U}^*)$$

where Γ is the familiar Gamma function. Using the integral representation of the Gamma and Beta functions, it can be shown that

$$Q_\beta^\gamma f(z) = \frac{1}{z} + \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+\gamma+1)} a_n z^n = \frac{1}{z} + \sum_{n=1}^{\infty} L(n,\beta,\gamma) a_n z^n$$

where

$$L(n,\beta,\gamma) = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)} \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+\gamma+1)}.$$

Let H be a complex Hilbert space and $L(H)$ denote the algebra of all bounded linear operators on H. For a complex-valued function f analytic in a domain E of the complex plain containing the spectrum $\sigma(T)$ of the bounded linear operator T, let $f(T)$ denote the operator on H defined by the Riesz-Dunford integral [1]

$$f(A) = \frac{1}{2\pi i} \int_C (zI-A)^{-1} f(z)dz,$$

where I is the identity operator on H and C is a positively oriented simple closed rectifiable closed contour containing the spectrum $\sigma(T)$ in the interior domain [3]. The operator $f(T)$ can also be defined by the following series:

$$f(T) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} T^n$$

which converges in the norm topology.

The class of all functions $f \in \Sigma$ with $a_n \geq 0$, is denoted by Σ_p. Now we introduce the following subclass of Σ_p associated with the integral operator $Q_\beta^\gamma f(z)$.

Definition 1. *For $0 \leq \delta < 1$ and $0 \leq \alpha < 1$, a function $f \in \Sigma_p$ given by (1) is in the class $M_p(\alpha,\delta,T)$ if*

$$\begin{aligned} &\left\| T(Q_\beta^\gamma f(T))' - \{(\delta-1)Q_\beta^\gamma f(T) + \delta T(Q_\beta^\gamma f(T))'\} \right\| \\ <\ &\left\| T(Q_\beta^\gamma f(T))' + (1-2\alpha)\{(\delta-1)Q_\beta^\gamma f(T) + \delta T(Q_\beta^\gamma f(T))'\} \right\| \end{aligned}$$

for all operators T with $\|T\| < 1$ and $T \neq \Theta$ (Θ is the zero operator on H).

In the present paper, we obtain coefficient estimates, radii of starlikeness, and convexity for the functions in the class $M_p(\alpha,\delta,T)$.

2. Coefficient Bounds

Theorem 1. *A function $f \in \Sigma_p$ given by* (1) *is in the class $M_p(\alpha, \delta, T)$ for all proper contraction T with $T \neq \Theta$ if and only if*

$$\sum_{n=1}^{\infty} [n + \alpha - \alpha\delta(n+1)] L(n, \beta, \gamma) a_n \leq 1 - \alpha. \tag{3}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{[n + \alpha - \alpha\delta(n+1)] L(n, \beta, \gamma)} z^n \quad (n \geq 1). \tag{4}$$

Proof. Assume that (3) holds. Then,

$$\begin{aligned}
& \left\| T(Q_\beta^\gamma f(T))' - \{(\delta - 1) Q_\beta^\gamma f(T) + \delta T (Q_\beta^\gamma f(T))'\} \right\| \\
& - \left\| T(Q_\beta^\gamma f(T))' + (1 - 2\alpha)\{(\delta - 1) Q_\beta^\gamma f(T) + \delta T (Q_\beta^\gamma f(T))'\} \right\| \\
= \ & \left\| \sum_{n=1}^{\infty} (n+1)(1-\delta) L(n, \beta, \gamma) a_n T^n \right\| \\
& - \left\| 2(1-\alpha) T^{-1} - \sum_{n=1}^{\infty} [n + (1-2\alpha)(\delta - 1 + \delta n)] L(n, \beta, \gamma) a_n T^n \right\| \\
\leq \ & \sum_{n=1}^{\infty} (n+1)(1-\delta) L(n, \beta, \gamma) a_n \|T\|^n - 2(1-\alpha) \left\| T^{-1} \right\| \\
& + \sum_{n=1}^{\infty} [n + (1-2\alpha)(\delta - 1 + \delta n)] L(n, \beta, \gamma) a_n \|T\|^n \\
= \ & 2 \sum_{n=1}^{\infty} [n + \alpha - \alpha\delta(n+1)] L(n, \beta, \gamma) a_n \|T\|^n - 2(1-\alpha) \left\| T^{-1} \right\| \\
\leq \ & 2(1-\alpha) - 2(1-\alpha) = 0, \qquad \text{(by using (3))}
\end{aligned}$$

and hence $f \in \Sigma_p$ is in the class $M_p(\alpha, \delta, T)$.
Conversely, let $f \in M_p(\alpha, \delta, T)$, that is,

$$\begin{aligned}
& \left\| T(Q_\beta^\gamma f(T))' - \{(\delta - 1) Q_\beta^\gamma f(T) + \delta T (Q_\beta^\gamma f(T))'\} \right\| \\
< \ & \left\| T(Q_\beta^\gamma f(T))' + (1 - 2\alpha)\{(\delta - 1) Q_\beta^\gamma f(T) + \delta T (Q_\beta^\gamma f(T))'\} \right\|.
\end{aligned}$$

From this inequality, it is obtained that

$$\left\|\sum_{n=1}^{\infty}(n+1)(1-\delta)L(n,\beta,\gamma)a_nT^{n+1}\right\|$$
$$< \left\|2(1-\alpha)-\sum_{n=1}^{\infty}[n+(1-2\alpha)(\delta-1+\delta n)]L(n,\beta,\gamma)a_nT^{n+1}\right\|.$$

By choosing $T=rI$ $(0<r<1)$ in above inequality, we get

$$\frac{\sum\limits_{n=1}^{\infty}(n+1)(1-\delta)L(n,\beta,\gamma)a_nr^{n+1}}{2(1-\alpha)-\sum\limits_{n=1}^{\infty}[n+(1-2\alpha)(\delta-1+\delta n)]L(n,\beta,\gamma)a_nr^{n+1}}<1.$$

As $r\to 1^-$, (3) is obtained.

Corollary 2. *If a function $f\in\Sigma_p$ given by* (1) *is in the class $M_p(\alpha,\delta,T)$, then*

$$a_n\leq\frac{1-\alpha}{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)}\quad(n\geq 1).$$

The result is sharp for the function f of the form (4).

Theorem 3. *The class $M_p(\alpha,\delta,T)$ is closed under convex combination.*

Proof. Let the functions

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n\qquad\text{and}\qquad g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}b_nz^n$$

be in the class $M_p(\alpha,\delta,T)$. Then, by Theorem 1, we have

$$\sum_{n=1}^{\infty}[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)a_n \leq 1-\alpha,$$
$$\sum_{n=1}^{\infty}[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)b_n \leq 1-\alpha.$$

For $0\leq\tau\leq 1$, define the function h as

$$h(z)=\tau f(z)+(1-\tau)g(z).$$

Then, we get

$$h(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\left[\tau a_n+(1-\tau)b_n\right]z^n.$$

Now, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)\left[\tau a_n+(1-\tau)b_n\right]\\
=\;&\tau\sum_{n=1}^{\infty}[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)a_n+(1-\tau)\sum_{n=1}^{\infty}[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)b_n\\
\leq\;&\tau(1-\alpha)+(1-\tau)(1-\alpha)\\
=\;&(1-\alpha).
\end{aligned}$$

So, $h\in M_p(\alpha,\delta,T)$.

3. Extreme points

Theorem 4. *Let*

$$f_0(z)=\frac{1}{z}$$

and

$$f_n(z)=\frac{1}{z}+\frac{1-\alpha}{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)}z^n\quad(n=1,2,\ldots).\tag{5}$$

Then $f\in M_p(\alpha,\delta,T)$ if and only if it can be represented in the form

$$f(z)=\sum_{n=0}^{\infty}\mu_n f_n(z)\quad\left(\mu_n\geq 0,\ \sum_{n=0}^{\infty}\mu_n=1\right).$$

Proof. Assume that $f(z)=\sum_{n=0}^{\infty}\mu_n f_n(z)$, $\left(\mu_n\geq 0,\ n=0,1,2,\ldots;\ \sum_{n=0}^{\infty}\mu_n=1\right)$. Then, we have

$$\begin{aligned}
f(z)&=\sum_{n=0}^{\infty}\mu_n f_n(z)\\
&=\mu_0 f_0(z)+\sum_{n=1}^{\infty}\mu_n f_n(z)\\
&=\frac{1}{z}+\sum_{n=1}^{\infty}\mu_n\frac{1-\alpha}{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)}z^n.
\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty}[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)\mu_n\frac{1-\alpha}{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)} &= (1-\alpha)\sum_{n=1}^{\infty}\mu_n \\ &= (1-\alpha)(1-\mu_0) \\ &\leq (1-\alpha).\end{aligned}$$

Hence, by Theorem 1, $f \in M_p(\alpha,\delta,T)$.
Conversely, suppose that $f \in M_p(\alpha,\delta,T)$. Since, by Corollary 2,

$$a_n \leq \frac{1-\alpha}{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)} \quad (n \geq 1)\,,$$

setting

$$\mu_n = \frac{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)}{1-\alpha}a_n \qquad (n \geq 1)$$

and $\mu_0 = 1 - \sum_{n=1}^{\infty}\mu_n$, we obtain

$$f(z) = \mu_0 f_0(z) + \sum_{n=1}^{\infty}\mu_n f_n(z).$$

This completes the proof of the theorem.

4. Radii of Starlikeness and Convexity

We now find the radii of meromorphically close-to-convexity, starlikeness and convexity for functions f in the class $M_p(\alpha,\delta,T)$.

Theorem 5. *Let $f \in M_p(\alpha,\delta,T)$. Then f is meromorphically close-to-convex of order μ $(0 \leq \mu < 1)$ in the disk $|z| < r_1$, where*

$$r_1 = \inf_n \left[\frac{(1-\mu)\,[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)}{n\,(1-\alpha)}\right]^{\frac{1}{n+1}} \qquad (n \geq 1)\,.$$

The result is sharp for the extremal function f given by (5).

Proof. It is sufficient to show that

$$\left\|\frac{f'(T)}{T^{-2}} + 1\right\| < 1-\mu. \tag{6}$$

By Theorem 1, we have

$$\sum_{n=1}^{\infty} \frac{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)}{1-\alpha} a_n \leq 1.$$

So the inequality

$$\left\| \frac{f'(T)}{T^{-2}} + 1 \right\| = \left\| \sum_{n=1}^{\infty} n a_n T^{n+1} \right\| \leq \sum_{n=1}^{\infty} n a_n \|T\|^{n+1} < 1-\mu$$

holds true if

$$\frac{n\|T\|^{n+1}}{1-\mu} \leq \frac{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)}{1-\alpha}.$$

Then, (6) holds true if

$$\|T\|^{n+1} \leq \frac{(1-\mu)\,[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)}{n\,(1-\alpha)} \qquad (n \geq 1)\,,$$

which yields the close-to-convexity of the family and completes the proof.

Theorem 6. *Let $f \in M_p(\alpha,\delta,T)$. Then f is meromorphically starlike of order μ $(0 \leq \mu < 1)$ in the disk $|z| < r_2$, where*

$$r_2 = \inf_n \left[\left(\frac{1-\mu}{n+2-\mu} \right) \frac{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)}{1-\alpha} \right]^{\frac{1}{n+1}} \qquad (n \geq 1)\,.$$

The result is sharp for the extremal function f given by (5).

Proof. By using the technique employed in the proof of Theorem 5, we can show that

$$\left\| \frac{Tf'(T)}{f(T)} + 1 \right\| < 1-\mu,$$

for $|z| < r_2$, and prove that the assertion of the theorem is true.

Theorem 7. *Let $f \in M_p(\alpha,\delta,T)$. Then f is meromorphically convex of order μ $(0 \leq \mu < 1)$ in the disk $|z| < r_3$, where*

$$r_3 = \inf_n \left[\left(\frac{1-\mu}{n+2-\mu} \right) \frac{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)}{n\,(1-\alpha)} \right]^{\frac{1}{n+1}} \qquad (n \geq 1)\,.$$

The result is sharp for the extremal function f given by

$$f_n(z) = \frac{1}{z} + \frac{n\,(1-\alpha)}{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)} z^n \qquad (n \geq 1)\,.$$

Proof. By using the technique employed in the proof of Theorem 5, we can show that

$$\left\|\frac{Tf''(T)}{f'(T)}+2\right\|<1-\mu,$$

for $|z|<r_3$, and prove that the assertion of the theorem is true.

5. Hadamard Product

Theorem 8. *For functions $f,g\in\Sigma_p$ defined by (1) and (2), respectively, let $f,g\in M_p(\alpha,\delta,T)$. Then the Hadamard product $f*g\in M_p(\rho,\delta,T)$, where*

$$\rho\leq\frac{[n+\alpha-\alpha\delta(n+1)]^2 L(n,\beta,\gamma)-n(1-\alpha)^2}{[n+\alpha-\alpha\delta(n+1)]^2 L(n,\beta,\gamma)+(1-\alpha)^2\left[1-\delta(n+1)\right]}.$$

Proof. From Theorem 1, we have

$$\sum_{n=1}^{\infty}\frac{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)}{1-\alpha}a_n \leq 1, \tag{7}$$

$$\sum_{n=1}^{\infty}\frac{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)}{1-\alpha}b_n \leq 1. \tag{8}$$

We need to find the largest ρ such that

$$\sum_{n=1}^{\infty}\frac{[n+\rho-\rho\delta(n+1)]L(n,\beta,\gamma)}{1-\rho}a_nb_n\leq 1.$$

From (7) and (8) we find, by means of the Cauchy-Schwarz inequality, that

$$\sum_{n=1}^{\infty}\frac{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)}{1-\alpha}\sqrt{a_nb_n}\leq 1. \tag{9}$$

Thus it is enough to show that

$$\frac{[n+\rho-\rho\delta(n+1)]L(n,\beta,\gamma)}{1-\rho}a_nb_n\leq\frac{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)}{1-\alpha}\sqrt{a_nb_n},$$

that is,

$$\sqrt{a_nb_n}\leq\frac{(1-\rho)\left[n+\alpha-\alpha\delta(n+1)\right]}{(1-\alpha)\left[n+\rho-\rho\delta(n+1)\right]}. \tag{10}$$

On the other hand, from (9) we have

$$\sqrt{a_n b_n} \leq \frac{1-\alpha}{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)}. \tag{11}$$

Therefore in view of (10) and (11) it is enough to show that

$$\frac{1-\alpha}{[n+\alpha-\alpha\delta(n+1)]L(n,\beta,\gamma)} \leq \frac{(1-\rho)\,[n+\alpha-\alpha\delta(n+1)]}{(1-\alpha)\,[n+\rho-\rho\delta(n+1)]}$$

which simplifies to

$$\rho \leq \frac{[n+\alpha-\alpha\delta(n+1)]^2\,L(n,\beta,\gamma)-n(1-\alpha)^2}{[n+\alpha-\alpha\delta(n+1)]^2\,L(n,\beta,\gamma)+(1-\alpha)^2\,[1-\delta(n+1)]}.$$

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