Acta Universitatis Apulensis No. 45/2016
ISSN: 1582-5329 pp. 61-72
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2016.45.05

# ON THE VERTEX-EDGE WIENER POLYNOMIALS OF THE DISJUNCTIVE PRODUCT OF GRAPHS

M. Azari and A. Iranmanesh

Abstract. In this paper, we study the behavior of the vertex-edge Wiener polynomials and their related indices under the disjunctive product of graphs. Results are applied to compute the vertex-edge Wiener indices for the disjunctive product of paths and cycles.

2010 *Mathematics Subject Classification*: 05C12, 05C76, 05C38.

*Keywords:* disjunctive product of graphs, vertex-edge Wiener index, vertex-edge Wiener polynomial.

## 1. Introduction

Throughout the paper, we consider connected finite graphs without any loops or multiple edges. A *topological index* (also known as *graph invariant*) is any function on a graph irrespective of the labeling of its vertices. Several hundreds of different invariants have been employed to date with various degrees of success in QSAR/QSPR studies. We refer the reader to monographs [11, 19] for review.

The oldest topological index is the one put forward in 1947 by Harold Wiener [20] nowadays referred to as the *Wiener index*. Wiener used his index for the calculation of the boiling points of alkanes. The Wiener index $W(G)$ of a graph $G$ is defined as the sum of distances between all pairs of vertices of $G$,

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u, v\,|G),$$

where $d(u, v\,|G)$ denotes the distance between the vertices $u$ and $v$ of $G$ which is the length of any shortest path in $G$ connecting $u$ and $v$. Details on the mathematical properties of the Wiener index and its applications can be found in [1, 2, 9, 10].

The *Hosoya polynomial* or *Wiener polynomial* [17] of a graph $G$ is defined in terms of a parameter $q$ as

$$W(G; q) = \sum_{\{u,v\}\subseteq V(G)} q^{d(u,v|G)}.$$

The first derivative of this polynomial at $q = 1$ is equal to the Wiener index, i.e., $W'(G;1) = W(G)$. We refer the reader to [12, 13, 14, 15] for more information on the Wiener polynomial.

In analogy with definition of the Wiener index, the vertex-edge versions of the Wiener index were defined based on distance between vertices and edges of a graph [8, 18]. Two possible distances between a vertex $u$ and an edge $e = ab$ of a graph $G$ can be considered. The first distance is denoted by $D_1(u, e\,|G)$ and defined as [18]

$$D_1(u, e\,|G) = \min\{d(u, a\,|G), d(u, b\,|G)\},$$

and the second one is denoted by $D_2(u, e\,|G)$ and defined as [8]

$$D_2(u, e\,|G) = \max\{d(u, a\,|G), d(u, b\,|G)\}.$$

Based on these two distances, two vertex-edge versions of the Wiener index can be introduced. The first and second *vertex-edge Wiener indices* of $G$ are denoted by $W_{ve_1}(G)$ and $W_{ve_2}(G)$, respectively, and defined as

$$W_{ve_i}(G) = \sum_{u\in V(G)} \sum_{e\in E(G)} D_i(u, e\,|G), \quad i \in \{1, 2\}.$$

The first and second *vertex-edge Wiener polynomials* of a graph $G$ are denoted by $W_{ve_1}(G;q)$ and $W_{ve_2}(G;q)$, respectively, and defined in terms of a parameter $q$ as [7]

$$W_{ve_i}(G;q) = \sum_{u\in V(G)} \sum_{e\in E(G)} q^{D_i(u,e|G)}, \quad i \in \{1, 2\}.$$

The first derivative of these polynomials at $q = 1$ are equal to their corresponding vertex-edge Wiener indices, i.e., $W'_{ve_i}(G;1) = W_{ve_i}(G)$, $i \in \{1, 2\}$.

For a graph $G$, let $N_G(u)$ denote the open neighborhood of a vertex $u$ in $G$ which is the set of all vertices of $G$ adjacent with $u$. The cardinality of $N_G(u)$ is called the degree of $u$ in $G$ and denoted by $d_G(u)$. One can easily see that,

$$\sum_{uv\in E(G)} |N_G(u) \cap N_G(v)| = 3\Delta(G),$$

where $\Delta(G)$ is the number of all triangles (3-cycles) in $G$. We denote by $N_G[u]$ the closed neighborhood of $u$ in $G$ which is defined as the set $N_G(u) \cup \{u\}$. If there is no ambiguity on $G$, we will omit the subscript $G$ in $N_G(u)$, $d_G(u)$, and $N_G[u]$.

The *first Zagreb index* of a graph $G$ is denoted by $M_1(G)$ and defined as [16]

$$M_1(G) = \sum_{u\in V(G)} d_G(u)^2.$$

The first Zagreb index can also be expressed by the following formulas,

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)],$$
$$M_1(G) = \sum_{u,v \in V(G)} |N(u) \cap N(v)|.$$

In [4, 8], the vertex-edge Wiener indices of some chemical graphs were computed and in [3, 5, 6, 7], the behavior of the vertex-edge Wiener indices and/or polynomials under some graph operations were investigated. In this paper, we compute the first and second vertex-edge Wiener polynomials and their related indices for the disjunctive product of graphs.

## 2. Results and discussion

Let $G_1$ and $G_2$ be two connected graphs. We denote by $V(G_i)$ and $E(G_i)$ the vertex set and edge set of $G_i$ and by $n_i$ and $e_i$ its order and size, respectively, where $i \in \{1, 2\}$. The *disjunctive product* $G_1 \vee G_2$ of graphs $G_1$ and $G_2$ is a graph with the vertex set $V(G_1) \times V(G_2)$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if and only if $u_1v_1 \in E(G_1)$ or $u_2v_2 \in E(G_2)$. The disjunctive product of two graphs is also known as their *co-normal product* or *OR product*. The distance between the vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $G_1 \vee G_2$ is given by

$$d(u, v\,|G_1 \vee G_2\,) = \begin{cases} 0 & if\ u_1 = v_1, u_2 = v_2, \\ 1 & if\ u_1v_1 \in E(G_1)\ or\ u_2v_2 \in E(G_2), \\ 2 & otherwise. \end{cases}$$

In this section, we compute the first and second vertex-edge Wiener polynomials and their related indices for the disjunctive product of $G_1$ and $G_2$. To do this, we first consider three subsets of $E(G_1 \vee G_2)$ as follows.

$$E_1 = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G_1), u_2, v_2 \in V(G_2)\},$$
$$E_2 = \{(u_1, u_2)(v_1, v_2) | u_2v_2 \in E(G_2), u_1, v_1 \in V(G_1)\},$$
$$E_3 = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G_1), u_2v_2 \in E(G_2)\}.$$

It is clear that, $E(G_1 \vee G_2) = \bigcup_{i=1}^{3} E_i$ and $|E(G_1 \vee G_2)| = e_1n_2^2 + e_2n_1^2 - 2e_1e_2$. Throughout the section, let $G = G_1 \vee G_2$.

## 2.1. The first vertex-edge Wiener polynomial and index

In this subsection, we compute the first vertex-edge Wiener polynomial and index for the disjunctive product of $G_1$ and $G_2$. At first, we prove some lemmas which will be used in the proof of our main theorem.

**Lemma 1.**

$$\begin{aligned}\sum_{u\in V(G)}\sum_{e\in E_1} q^{D_1(u,e|G)} &= 2e_1n_2^2 + \Big[n_2^3\big(M_1(G_1)-3\Delta(G_1)\big) - 2e_1n_2^2 \\ &+ \big(n_1e_1 - M_1(G_1)+3\Delta(G_1)\big)\big(4n_2e_2 - M_1(G_2)\big)\Big]q \\ &+ \big(n_1e_1 - M_1(G_1)+3\Delta(G_1)\big)\big(n_2^3 - 4n_2e_2 + M_1(G_2)\big)q^2.\end{aligned} \tag{1}$$

*Proof.* Let $A = \sum_{u\in V(G)}\sum_{e\in E_1} q^{D_1(u,e|G)}$. By definition of the set $E_1$, we have

$$\begin{aligned} A &= \sum_{(u_1,u_2)\in V(G)}\ \sum_{(a_1,a_2)(b_1,b_2)\in E_1} q^{\min\{d((u_1,u_2),(a_1,a_2)|G),d((u_1,u_2),(b_1,b_2)|G)\}} \\ &= \sum_{a_1b_1\in E(G_1)}\ \sum_{a_2\in V(G_2)}\ \sum_{b_2\in V(G_2)} \Big[\sum_{u_1=a_1}\sum_{u_2=a_2} q^0 + \sum_{u_1=b_1}\sum_{u_2=b_2} q^0 \\ &+ \sum_{u_1=a_1}\ \sum_{u_2\in V(G_2)-\{a_2\}} q^1 + \sum_{u_1=b_1}\ \sum_{u_2\in V(G_2)-\{b_2\}} q^1 \\ &+ \sum_{u_1\in (N(a_1)\cup N(b_1))-\{a_1,b_1\}}\ \sum_{u_2\in V(G_2)} q^1 + \sum_{u_1\in V(G_1)-(N(a_1)\cup N(b_1))}\ \sum_{u_2\in N(a_2)\cup N(b_2)} q^1 \\ &+ \sum_{u_1\in V(G_1)-(N(a_1)\cup N(b_1))}\ \sum_{u_2\in V(G_2)-(N(a_2)\cup N(b_2))} q^2\Big] \\ &= 2e_1n_2^2 + \Big[2e_1n_2^2(n_2-1) + n_2^3 \sum_{a_1b_1\in E(G_1)}\big(d(a_1)+d(b_1)-|N(a_1)\cap N(b_1)|-2\big) \\ &+ \sum_{a_1b_1\in E(G_1)}\big(n_1 - d(a_1) - d(b_1) + |N(a_1)\cap N(b_1)|\big) \\ &\sum_{a_2\in V(G_2)}\ \sum_{b_2\in V(G_2)}\big(d(a_2)+d(b_2)-|N(a_2)\cap N(b_2)|\big)\Big]q \\ &+ \sum_{a_1b_1\in E(G_1)}\big(n_1 - d(a_1) - d(b_1) + |N(a_1)\cap N(b_1)|\big) \\ &\sum_{a_2\in V(G_2)}\ \sum_{b_2\in V(G_2)}\big(n_2 - d(a_2) - d(b_2) + |N(a_2)\cap N(b_2)|\big)q^2. \end{aligned}$$

Eq. (1) is obtained after simplifying the above expression.

**Lemma 2.**

$$\sum_{u\in V(G)}\sum_{e\in E_2} q^{D_1(u,e|G)} = 2e_2n_1^2 + \Big[n_1^3\big(M_1(G_2)-3\Delta(G_2)\big) - 2e_2n_1^2 + \big(n_2e_2 - M_1(G_2)+3\Delta(G_2)\big)\big(4n_1e_1 - M_1(G_1)\big)\Big]q + \big(n_2e_2 - M_1(G_2)+3\Delta(G_2)\big)\left(n_1^3 - 4n_1e_1 + M_1(G_1)\right)q^2. \quad (2)$$

*Proof.* The proof is similar to the proof of Lemma 1.

**Lemma 3.**

$$\sum_{u\in V(G)}\sum_{e\in E_3} q^{D_1(u,e|G)} = 2e_1e_2 + \Big[n_2e_2\big(M_1(G_1)-3\Delta(G_1)\big) - 2e_1e_2 + \big(n_1e_1 - M_1(G_1)+3\Delta(G_1)\big)\big(M_1(G_2) - 3\Delta(G_2)\big)\Big]q + \big(n_1e_1 - M_1(G_1)+3\Delta(G_1)\big)\big(n_2e_2 - M_1(G_2) + 3\Delta(G_2)\big)q^2. \quad (3)$$

*Proof.* Let $C = \sum_{u\in V(G)}\sum_{e\in E_3} q^{D_1(u,e|G)}$. By definition of the set $E_3$, we have

$$\begin{aligned}
C =& \sum_{(u_1,u_2)\in V(G)}\ \sum_{(a_1,a_2)(b_1,b_2)\in E_3} q^{\min\{d((u_1,u_2),(a_1,a_2)|G),d((u_1,u_2),(b_1,b_2)|G)\}}\\
=& \sum_{a_1b_1\in E(G_1)}\ \sum_{a_2b_2\in E(G_2)}\Big[\sum_{u_1=a_1}\sum_{u_2=a_2} q^0 + \sum_{u_1=b_1}\sum_{u_2=b_2} q^0 + \sum_{u_1=a_1}\ \sum_{u_2\in V(G_2)-\{a_2\}} q^1\\
&+ \sum_{u_1=b_1}\ \sum_{u_2\in V(G_2)-\{b_2\}} q^1 + \sum_{u_1\in(N(a_1)\cup N(b_1))-\{a_1,b_1\}}\ \sum_{u_2\in V(G_2)} q^1\\
&+ \sum_{u_1\in V(G_1)-(N(a_1)\cup N(b_1))}\ \sum_{u_2\in N(a_2)\cup N(b_2)} q^1\\
&+ \sum_{u_1\in V(G_1)-(N(a_1)\cup N(b_1))}\ \sum_{u_2\in V(G_2)-(N(a_2)\cup N(b_2))} q^2\Big]\\
=&2e_1e_2 + \Big[2e_1e_2(n_2-1) + n_2e_2\sum_{a_1b_1\in E(G_1)}\big(d(a_1)+d(b_1) - |N(a_1)\cap N(b_1)| - 2\big)\\
&+ \sum_{a_1b_1\in E(G_1)}\big(n_1 - d(a_1) - d(b_1) + |N(a_1)\cap N(b_1)|\big)\\
&\sum_{a_2b_2\in E(G_2)}\big(d(a_2)+d(b_2) - |N(a_2)\cap N(b_2)|\big)\Big]q\\
&+ \sum_{a_1b_1\in E(G_1)}\big(n_1 - d(a_1) - d(b_1) + |N(a_1)\cap N(b_1)|\big)
\end{aligned}$$

$$\sum_{a_2b_2\in E(G_2)} \big(n_2 - d(a_2) - d(b_2) + |N(a_2)\cap N(b_2)|\big)q^2.$$

Eq. (3) is obtained after simplifying the above expression.

Let $e = (a_1, a_2)(b_1, b_2)$ be an edge of $G$ which belongs to $E_3$. Then, obviously, $(a_1, b_2)(b_1, a_2)$ is also an edge in $E_3$. We denote the edge $(a_1, b_2)(b_1, a_2)$ by $\bar{e}$.

**Lemma 4.**

$$\sum_{u\in V(G)}\sum_{e\in E_3} q^{D_1(u,e|G)} = \sum_{u\in V(G)}\sum_{e\in E_3} q^{D_1(u,\bar{e}|G)}. \tag{4}$$

*Proof.* The formula of $\sum_{u\in V(G)}\sum_{e\in E_3} q^{D_1(u,\bar{e}|G)}$ can easily be obtained by changing the role of the vertices $a_2$ and $b_2$ in the proof of Lemma 3. On the other hand, one can easily check that, changing the role of $a_2$ and $b_2$ in the proof of Lemma 3 does not influence the obtained result. So, Eq. (4) holds.

Now, we are ready to compute the first vertex-edge Wiener polynomial of the disjunctive product of $G_1$ and $G_2$.

**Theorem 5.** *The first vertex-edge Wiener polynomial of the disjunctive product of $G_1$ and $G_2$ is given by*

$$\begin{aligned}W_{ve_1}(G;q) =& 2e_1(n_2^2 - e_2) + 2e_2(n_1^2 - e_1) + \big[4e_1e_2(2n_1n_2 + 1) - 2(e_1n_2^2 + e_2n_1^2)\\ &+ (n_2^3 - 7n_2e_2)M_1(G_1) + (n_1^3 - 7n_1e_1)M_1(G_2) - 3(n_2^3 - 6n_2e_2)\Delta(G_1)\\ &- 3(n_1^3 - 6n_1e_1)\Delta(G_2) - 9\Delta(G_1)M_1(G_2) - 9\Delta(G_2)M_1(G_1)\\ &+4M_1(G_1)M_1(G_2) + 18\Delta(G_1)\Delta(G_2)\big]\, q + [n_1\, n_2e_2(n_2^2 - 5e_2)\\ &+ n_1n_2e_2(n_1^2 - 5e_1) - (n_2^3 - 7n_2e_2)M_1(G_1) - (n_1^3 - 7n_1e_1)M_1(G_2)\\ &+ 3(n_2^3 - 6n_2e_2)\Delta(G_1) + 3(n_1^3 - 6n_1e_1)\Delta(G_2) + 9\Delta(G_1)M_1(G_2)\\ &+ 9\Delta(G_2)M_1(G_1) - 4M_1(G_1)M_1(G_2) - 18\Delta(G_1)\Delta(G_2)]\, q^2.\end{aligned} \tag{5}$$

*Proof.* By definitions of the first vertex-edge Wiener polynomial and disjunctive product, we have

$$\begin{aligned}W_{ve_1}(G;q) = &\sum_{u\in V(G)}\sum_{e\in E_1} q^{D_1(u,e|G)} + \sum_{u\in V(G)}\sum_{e\in E_2} q^{D_1(u,e|G)}\\ &- \sum_{u\in V(G)}\sum_{e\in E_3} (q^{D_1(u,e|G)} + q^{D_1(u,\bar{e}|G)}).\end{aligned}$$

By Eq. (4),

$$W_{ve_1}(G;q) = \sum_{u\in V(G)}\sum_{e\in E_1} q^{D_1(u,e|G)} + \sum_{u\in V(G)}\sum_{e\in E_2} q^{D_1(u,e|G)} - 2\sum_{u\in V(G)}\sum_{e\in E_3} q^{D_1(u,e|G)}.$$

Now, using Eqs. (1), (2), and (3), we can get Eq. (5).

By taking the first derivative from Eq. (5) with respect to $q$, and then by substituting $q = 1$, we can easily obtain a formula for the first vertex-edge Wiener index of the disjunctive product of $G_1$ and $G_2$.

**Corollary 6.** *The first vertex-edge Wiener index of the disjunctive product of $G_1$ and $G_2$ is given by*

$$\begin{aligned} W_{ve_1}(G) =& 2(e_1n_2^2 + e_2n_1^2)(n_1n_2 - 1) - 4e_1e_2(3n_1n_2 - 1) - (n_2^3 - 7n_2e_2)M_1(G_1) \\ & -(n_1^3 - 7n_1e_1)M_1(G_2) + 3(n_2^3 - 6n_2e_2)\Delta(G_1) + 3(n_1^3 - 6n_1e_1)\Delta(G_2) \\ & +9\Delta(G_1)M_1(G_2) + 9\Delta(G_2)M_1(G_1) - 4M_1(G_1)M_1(G_2) - 18\Delta(G_1)\Delta(G_2). \end{aligned} \tag{6}$$

Let $P_n$ and $C_n$ denote the $n$-vertex path and cycle, respectively. It is easy to see that, for $n \geq 2$, $M_1(P_n) = 4n - 6$ and for $n \geq 3$, $M_1(C_n) = 4n$. Also $\Delta(P_n) = 0$, $\Delta(C_3) = 1$, and for $n \geq 4$, $\Delta(C_n) = 0$. Now, using Eq. (6), we easily arrive at:

**Corollary 7.** *For every positive integers $n \geq 2$ and $m \geq 3$,*

$$W_{ve_1}(P_n \vee C_m) = \begin{cases} 9n^3 + 6n^2 - 30n + 24 & if\ m = 3, \\ 2m^3(n^2 - 3n + 3) + 2m^2(n^3 - 6n^2 + 19n - 20) & if\ m \geq 4 \\ -2m(2n^3 - 13n^2 + 44n - 46). & \end{cases}$$

## 2.2. The second vertex-edge Wiener polynomial and index

In this subsection, we compute the second vertex-edge Wiener polynomial and index for the disjunctive product of $G_1$ and $G_2$. At first, we prove some lemmas which will be used in the proof of our main theorem.

**Lemma 8.**

$$\begin{aligned} \sum_{u\in V(G)}\sum_{e\in E_1} q^{D_2(u,e|G)} =& \big[2e_1n_2^2 + 2n_2e_2M_1(G_1) + 3(n_2^3 - 4n_2e_2)\Delta(G_1) + \big(n_1e_1 \\ & -M_1(G_1) + 3\Delta(G_1)\big)M_1(G_2)\big]\, q + \big[n_1e_1n_2^3 - 2e_1n_2^2 - 2n_2e_2M_1(G_1) \\ & + 3(4n_2e_2 - n_2^3)\Delta(G_1) - \big(n_1e_1 - M_1(G_1) + 3\Delta(G_1)\big)M_1(G_2)\,\big]\, q^2. \end{aligned} \tag{7}$$

*Proof.* Let $A' = \sum_{u\in V(G)} \sum_{e\in E_1} q^{D_2(u,e|G)}$. By definition of the set $E_1$, we have

$$\begin{aligned}
A' &= \sum_{(u_1,u_2)\in V(G)} \sum_{(a_1,a_2)(b_1,b_2)\in E_1} q^{\max\{d((u_1,u_2),(a_1,a_2)|G),d((u_1,u_2),(b_1,b_2)|G)\}}\\
&= \sum_{a_1b_1\in E(G_1)} \sum_{a_2\in V(G_2)} \sum_{b_2\in V(G_2)} \Big[\sum_{u_1=a_1} \sum_{u_2\in N[a_2]} q^1 + \sum_{u_1=b_1} \sum_{u_2\in N[b_2]} q^1\\
&\quad + \sum_{u_1\in N(a_1)-N[b_1]} \sum_{u_2\in N(b_2)} q^1 + \sum_{u_1\in N(b_1)-N[a_1]} \sum_{u_2\in N(a_2)} q^1\\
&\quad + \sum_{u_1\in N(a_1)\cap N(b_1)} \sum_{u_2\in V(G_2)} q^1 + \sum_{u_1\in V(G_1)-(N(a_1)\cup N(b_1))} \sum_{u_2\in N(a_2)\cap N(b_2)} q^1\\
&\quad + \sum_{u_1=a_1} \sum_{u_2\in V(G_2)-N[a_2]} q^2 + \sum_{u_1=b_1} \sum_{u_2\in V(G_2)-N[b_2]} q^2\\
&\quad + \sum_{u_1\in N(a_1)-N[b_1]} \sum_{u_2\in V(G_2)-N(b_2)} q^2 + \sum_{u_1\in N(b_1)-N[a_1]} \sum_{u_2\in V(G_2)-N(a_2)} q^2\\
&\quad + \sum_{u_1\in V(G_1)-(N(a_1)\cup N(b_1))} \sum_{u_2\in V(G_2)-(N(a_2)\cap N(b_2))} q^2\Big].
\end{aligned}$$

Eq. (7) is obtained after simplifying the above expression.

**Lemma 9.**

$$\begin{aligned}
\sum_{u\in V(G)} \sum_{e\in E_2} q^{D_2(u,e|G)} &= \big[2e_2n_1^2 + 2n_1e_1M_1(G_2) + 3\big(n_1^3 - 4n_1e_1\big)\Delta(G_2) + \big(n_2e_2\\
&\quad - M_1(G_2) + 3\Delta(G_2)\big)M_1(G_1)\,\big]\,q + \big[n_2e_2n_1^3 - 2e_2n_1^2 - 2n_1e_1M_1(G_2)\\
&\quad + 3\big(4n_1e_1 - n_1^3\big)\Delta(G_2) - \big(n_2e_2 - M_1(G_2) + 3\Delta(G_2)\big)M_1(G_1)\big]\,q^2.
\end{aligned} \tag{8}$$

*Proof.* The proof is similar to the proof of Lemma 8.

**Lemma 10.**

$$\begin{aligned}
\sum_{u\in V(G)} \sum_{e\in E_3} (q^{D_2(u,e|G)} + q^{D_2(u,\bar{e}|G)}) &= [4e_1e_2 + 6n_2e_2\Delta(G_1) + 6n_1e_1\Delta(G_2)\\
&\quad - 6\Delta(G_1)M_1(G_2) - 6\Delta(G_2)M_1(G_1) + M_1(G_1)M_1(G_2) + 18\Delta(G_1)\Delta(G_2\ )]\,q\\
&\quad + [2e_1e_2(n_1n_2 - 2) + 6M_1(G_1)\Delta(G_2) + 6M_1(G_2)\Delta(G_1) - 6n_1e_1\Delta(G_2)\\
&\quad - 6n_2e_2\Delta(G_1) - 18\Delta(G_1)\Delta(G_2) - M_1(G_1)M_1(G_2\ )]\,q^2.
\end{aligned} \tag{9}$$

*Proof.* Let $C' = \sum_{u\in V(G)} \sum_{e\in E_3} q^{D_2(u,e|G)}$, and $\bar{C}' = \sum_{u\in V(G)} \sum_{e\in E_3} q^{D_2(u,e|G)}$. By definition of the set $E_3$, we have

$$\begin{aligned}
C' &= \sum_{(u_1,u_2)\in V(G)} \sum_{(a_1,a_2)(b_1,b_2)\in E_3} q^{\max\{d((u_1,u_2),(a_1,a_2)|G),d((u_1,u_2),(b_1,b_2)|G)\}} \\
&= \sum_{a_1b_1\in E(G_1)} \sum_{a_2b_2\in E(G_2)} \Big[ \sum_{u_1=a_1} \sum_{u_2\in N[a_2]} q^1 + \sum_{u_1=b_1} \sum_{u_2\in N[b_2]} q^1 \\
&+ \sum_{u_1\in N(a_1)-N[b_1]} \sum_{u_2\in N(b_2)} q^1 + \sum_{u_1\in N(b_1)-N[a_1]} \sum_{u_2\in N(a_2)} q^1 \\
&+ \sum_{u_1\in N(a_1)\cap N(b_1)} \sum_{u_2\in V(G_2)} q^1 + \sum_{u_1\in V(G_1)-(N(a_1)\cup N(b_1))} \sum_{u_2\in N(a_2)\cap N(b_2)} q^1 \\
&+ \sum_{u_1=a_1} \sum_{u_2\in V(G_2)-N[a_2]} q^2 + \sum_{u_1=b_1} \sum_{u_2\in V(G_2)-N[b_2]} q^2 \\
&+ \sum_{u_1\in N(a_1)-N[b_1]} \sum_{u_2\in V(G_2)-N(b_2)} q^2 + \sum_{u_1\in N(b_1)-N[a_1]} \sum_{u_2\in V(G_2)-N(a_2)} q^2 \\
&+ \sum_{u_1\in V(G_1)-(N(a_1)\cup N(b_1))} \sum_{u_2\in V(G_2)-(N(a_2)\cap N(b_2))} q^2 \Big].
\end{aligned}$$

The formula of $\bar{C}'$ can easily be obtained by changing the role of the vertices $a_2$ and $b_2$ in the formula of $C'$. Now, Eq. (9) is obtained by adding the formulas of $C'$ and $\bar{C}'$ and simplifying the resulting expression.

Now, we are ready to compute the second vertex-edge Wiener polynomial of the disjunctive product of $G_1$ and $G_2$.

**Theorem 11.** *The second vertex-edge Wiener polynomial of the disjunctive product of $G_1$ and $G_2$ is given by*

$$\begin{aligned}
W_{ve_2}(G;q) = &[2\, e_1(n_2^2 - e_2) + 2e_2(n_1^2 - e_1) + 3n_2e_2M_1(G_1) + 3n_1e_1M_1(G_2) \\
&+ 3(n_2^3 - 6n_2e_2)\Delta(G_1) + 3(n_1^3 - 6n_1e_1)\Delta(G_2) + 9\Delta(G_2)M_1(G_1) \\
&+ 9\Delta(G_1)M_1(G_2) - 3M_1(G_1)M_1(G_2) - 18\Delta(G_1)\Delta(G_2)]\, q \\
&+ [(n_1n_2 - 2)(e_1n_2^2 + e_2n_1^2 - 2e_1e_2) - 3n_2e_2M_1(G_1) - 3n_1e_1M_1(G_2) \\
&- 3(n_2^3 - 6n_2e_2)\Delta(G_1) - 3(n_1^3 - 6n_1e_1)\Delta(G_2) - 9M_1(G_1)\Delta(G_2) \\
&- 9M_1(G_2)\Delta(G_1) + 3M_1(G_1)M_1(G_2) + 18\Delta(G_1)\Delta(G_2)]\, q^2.
\end{aligned} \tag{10}$$

*Proof.* By definitions of the second vertex-edge Wiener polynomial and disjunctive product, we have

$$W_{ve_2}(G;q) = \sum_{u\in V(G)}\sum_{e\in E_1} q^{D_2(u,e|G)} + \sum_{u\in V(G)}\sum_{e\in E_2} q^{D_2(u,e|G)} - \sum_{u\in V(G)}\sum_{e\in E_3} (q^{D_2(u,e|G)} + q^{D_2(u,\bar{e}|G)}).$$

Now, by Eqs. (7), (8), and (9), we can get Eq. (10).

By taking the first derivative from Eq. (10) with respect to $q$, and then by substituting $q = 1$, we can easily obtain a formula for the second vertex-edge Wiener index of the disjunctive product of $G_1$ and $G_2$.

**Corollary 12.** *The second vertex-edge Wiener index of the disjunctive product of $G_1$ and $G_2$ is given by*

$$\begin{aligned} W_{ve_2}(G) =& 2(n_1n_2-1)(e_1n_2^2+e_2n_1^2-2e_1e_2) - 3n_2e_2M_1(G_1) - 3n_1e_1M_1(G_2) \\ & - 3(n_2^3-6n_2e_2)\Delta(G_1) - 3(n_1^3-6n_1e_1)\Delta(G_2) - 9M_1(G_1)\Delta(G_2) \\ & - 9M_1(G_2)\Delta(G_1) + 3M_1(G_1)M_1(G_2) + 18\Delta(G_1)\Delta(G_2). \end{aligned} \tag{11}$$

As a direct consequence of Eq. (11), we easily arrive at:

**Corollary 13.** *For every positive integers $n \geq 2$ and $m \geq 3$,*

$$W_{ve_2}(P_n \vee C_m) = \begin{cases} 15n^3 - 6n^2 - 6n + 6 & if\ m = 3, \\ 2m^3n(n-1) + 2m^2(n^3 - 2n^2 - 5n + 10) & if\ m \geq 4 \\ -2m(7n^2 - 32n + 38). & \end{cases}$$

**Acknowledgements.** Partial support by the Center of Excellence of Algebraic Hyper-structures and its Applications of Tarbiat Modares University (CEAHA) is gratefully acknowledged by the second author (AI).

## References

[1] A. R. Ashrafi, Z. Mohammad-Abadi, *On Wiener index of one-heptagonal nanocone*, Curr. Nanosci. 8, 1 (2012), 180-185.

[2] A. R. Ashrafi, A. Karbasioun, M. V. Diudea, *Computing Wiener and detour indices of a new type of nanostar dendrimers*, MATCH Commun. Math. Comput. Chem. 65 (2011), 193-200.

[3] M. Azari, *A note on vertex-edge Wiener indices*, Iranian J. Math. Chem. 7, 1 (2016), 11-17.

[4] M. Azari, A. Iranmanesh, *Computation of the edge Wiener indices of the sum of graphs*, Ars Combin. 100 (2011), 113-128.

[5] M. Azari, A. Iranmanesh, *Clusters and various versions of Wiener-type invariants*, Kragujevac J. Math. 39, 2 (2015), 155-171.

[6] M. Azari, A. Iranmanesh, *Joins, coronas and their vertex-edge Wiener polynomials*, Tamkang J. Math. (to appear)

[7] M. Azari, A. Iranmanesh, A. Tehranian, *Maximum and minimum polynomials of a composite graph*, Austral. J. Basic Appl. Sci. 5, 9 (2011), 825-830.

[8] M. Azari, A. Iranmanesh, A. Tehranian, *Two topological indices of three chemical structures*, MATCH Commun. Math. Comput. Chem. 69 (2013), 69-86.

[9] M. R. Darafsheh, M. H. Khalifeh, *Calculation of the Wiener, Szeged, and PI indices of a certain nanostar dendrimer*, Ars Combin. 100 (2011), 289-298.

[10] M. V. Diudea, *Wiener index of dendrimers*, MATCH Commun. Math. Comput. Chem. 32 (1995), 71–83.

[11] M. V. Diudea, *QSPR/QSAR Studies by Molecular Descriptors*, NOVA, New York (2001).

[12] M. V. Diudea, *Hosoya polynomial in tori*, MATCH Commun. Math. Comput. Chem. 45 (2002), 109–122.

[13] T. Došlić, *Splices, links and their degree-weighted Wiener polynomials*, Graph Theory Notes N. Y. 48 (2005), 47-55.

[14] T. Došlić, *Vertex-weighted Wiener polynomials for composite graphs*, Ars Math. Contemp. 1, 1 (2008), 66-80.

[15] M. Eliasi, A. Iranmanesh, *Hosoya polynomial of hierarchical product of graphs*, MATCH Commun. Math. Comput. Chem. 69 (2013), 111–119.

[16] I. Gutman, N. Trinajstić, *Graph theory and molecular orbitals, Total $\pi-$electron energy of alternant hydrocarbons*, Chem. Phys. Lett. 17 (1972), 535-538.

[17] H. Hosoya, *On some counting polynomials in chemistry*, Discrete Appl. Math. 19 (1988), 239–257.

[18] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, S. G. Wagner, *Some new results on distance-based graph invariants*, European J. Combin. 30 (2009), 1149–1163.

[19] N. Trinajstić, *Chemical Graph Theory*, CRC Press, Boca Raton (1992).

[20] H. Wiener, *Structural determination of paraffin boiling points*, J. Am. Chem. Soc. 69 (1947), 17–20.

Mahdieh Azari
Department of Mathematics,
Kazerun Branch, Islamic Azad University
P. O. Box: 73135-168, Kazerun, Iran
email: *azari@kau.ac.ir, mahdie.azari@gmail.com*

Ali Iranmanesh
Department of Mathematics, Faculty of Mathematical Sciences,
Tarbiat Modares University
P. O. Box: 14115-137, Tehran, Iran
email: *iranmanesh@modares.ac.ir*