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A REMARK ON A SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract. In the present paper, we investigate a subclass of analytic functions defined by a multiplier transformation. The class is previously studied by Laura Stanciu and Daniel Breaz [6] for $f \in \mathcal{A}$. We here study this class for $f \in \mathcal{A}_p$ and obtained certain results for starlikeness and convexity of analytic functions $f \in \mathcal{A}_p$. We also present correct version of some results of Laura Stanciu and Daniel Breaz [6].

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1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of all functions f which are analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions that $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathcal{A}$ has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \ , \quad p \in \mathbb{N} = \{1, 2, 3 \cdots\},$$

analytic and multivalent in the open unit disk $\mathbb{E}$. Note that $\mathcal{A}_1 = \mathcal{A}$. For $f \in \mathcal{A}_p$, define the multiplier transformation $I_p(n, \lambda)$ on class $\mathcal{A}_p$ as

$$I_p(n, \lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k, \quad (\lambda \geq 0, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

The transformation $I_p(n,\lambda)$ has been recently studied by Aghalary [1], Billing ([2], [3], [4], [5]), Singh et al.[11]. The special case $I_1(n,0)$ of above defined transformation is the well-known Sălăgean [10] derivative operator D^n, defined for $f\in\mathcal{A}$ as under:

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

A function $f\in\mathcal{A}_p$ is said to be p-valent starlike of order α $(0\le\alpha<p)$ in $\mathbb{E}$, if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z\in\mathbb{E}.$$

Let $\mathcal{S}_p^*(\alpha)$ denote the class of all p-valent starlike functions of order α $(0\le\alpha<p)$.
A function $f\in A_p$ is said to be p-valent convex of order α $(0\le\alpha<p)$ in $\mathbb{E}$ if it satisfies the condition

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha,\ z\in\mathbb{E}.$$

We denote by $\mathcal{K}_p(\alpha)$, the class of all functions $f\in\mathcal{A}_p$ that are p-valent convex of order α $(0\le\alpha<p)$ in $\mathbb{E}$. Note that $\mathcal{S}^*(\alpha)=\mathcal{S}_1^*(\alpha)$ and $\mathcal{K}(\alpha)=\mathcal{K}_1(\alpha)$ are the usual classes of univalent starlike functions (w.r.t. the origin) of order α $(0\le\alpha<1)$and univalent convex functions of order α $(0\le\alpha<1)$.
A function $f\in\mathcal{A}$ is said to be close-to-convex of order $\alpha,\ 0\le\alpha<1$ in $\mathbb{E}$ if

$$\Re\left(\frac{f'(z)}{g'(z)}\right) > \alpha,\ z\in\mathbb{E}, \tag{1}$$

for a convex function g (not necessarily normalized). The class of close-to-convex functions of order α is denoted by $\mathcal{C}(\alpha)$. Let $\mathcal{C}=\mathcal{C}(0)$ denote the class of close-to-convex functions. It is well-known that every close-to-convex function is univalent. In case $\alpha=0$ and $g(z)\equiv z$, the condition (1) reduces to

$$\Re\ f'(z)>0,\ z\in\mathbb{E} \quad\Rightarrow\quad f\in\mathcal{C}.$$

This simple but elegant result was independently proved by Noshiro [8] and Warchawski [12] in 1934/35.
For two analytic functions f and g in the unit disk $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$ and write as $f\prec g$ if there exists a Schwarz function w analytic in $\mathbb{E}$ with $w(0)=0$ and $|w(z)|<1,\ z\in\mathbb{E}$ such that $f(z)=g(w(z)),\ z\in\mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to: $f(0)=g(0)$ and $f(\mathbb{E})\subset g(\mathbb{E})$.

Obradovič [9] introduced and studied the class $\mathcal{N}(\alpha), 0 < \alpha < 1$ of functions $f \in \mathcal{A}$ satisfying the following inequality

$$\Re\left\{f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha}\right\} > 0, \quad z \in \mathbb{E}.$$

He obtained the starlikeness of members of $\mathcal{N}(\alpha)$. We, here, define below a more general class involving the multiplier transformation $I_p(n, \lambda)$.
A function $f \in \mathcal{A}_p$ is in the class $B_p(n, \lambda, \mu, \alpha), \quad n \in N, \quad \mu \geq 0, \quad \alpha \in [0, 1)$ if

$$\left|\frac{I_p(n+1, \lambda)f(z)}{z}\left(\frac{z}{I_p(n, \lambda)f(z)}\right)^{\mu} - 1\right| < 1 - \alpha, \quad z \in \mathbb{E}.$$

The operator $I_1(n, \lambda)$ is recently studied by Laura Stanciu and Daniel Breaz [6]. They obtained certain sufficient conditions for $f \in \mathcal{A}$ to be a member of the class $B(n, \lambda, \mu, \alpha) = B_1(n, \lambda, \mu, \alpha)$.
The main objective of this paper is to find certain sufficient conditions for $f \in \mathcal{A}_p$ to be a member of $B_p(n, \lambda, \mu, \alpha)$ and consequently, we get certain criteria for starlikeness and convexity of analytic functions $f \in \mathcal{A}_p$. At the same time, we also present the correct version of the results obtained by Laura Stanciu and Daniel Breaz [6]. In fact, we point out the following difficulties while deriving the results of Laura Stanciu and Daniel Breaz [6].

1. We could not verified the equality in equation (2) of Laura Stanciu and Daniel Breaz [6].

2. We could not understand how to arrive at the condition

$$\Re\left(1 + \frac{zu'(z)}{u(z)}\right) > \frac{3\alpha - 1}{2\alpha},$$

in the proof of main result Theorem 1 of Laura Stanciu and Daniel Breaz [6].

3. We also notice that

$$\Re\left(\frac{I(n+1, \lambda)f(z)}{z}\left(\frac{z}{I(n, \lambda)f(z)}\right)^{\mu}\right) > \alpha.$$

does not imply that $f \in B(n, \lambda, \mu, \alpha)$ because the above condition does not imply

$$\left|\frac{I_p(n+1, \lambda)f(z)}{z}\left(\frac{z}{I_p(n, \lambda)f(z)}\right)^{\mu} - 1\right| < 1 - \alpha.$$

4. We also notice that f should be a member of $\mathcal{A}$ in place of $\mathcal{A}_p$ in both Definition 2 and Theorem 1 of Laura Stanciu and Daniel Breaz [6].

To prove our main result, we shall use the following lemma of Miller and Mocanu [7, page 76].

Lemma 1. *Let h be starlike in $\mathbb{E}$ with $h(0) = 0$. If an analytic function $p(z) \neq 0$ in $\mathbb{E}$ satisfies*

$$\frac{zp'(z)}{p(z)} \prec h(z), \ z \in \mathbb{E},$$

then

$$p(z) \prec q(z) = \exp\left[\int_0^z \frac{h(t)}{t} dt\right]$$

and q is the best dominant.

2. Main Result

Theorem 2. *Let μ be a real number such that $\mu \geq 0$. If $f \in \mathcal{A}_p$ satisfies the condition*

$$(p+\lambda)\left\{(\mu-1) + \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} - \mu\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right\} \prec \frac{(1-\alpha)z}{1+(1-\alpha)z}, \ 0 \leq \alpha < 1, \tag{2}$$

then

$$f \in B_p(n, \lambda, \mu, \alpha).$$

Proof. On writing

$$\frac{I_p(n+1,\lambda)f(z)}{z^p}\left(\frac{z^p}{I_p(n,\lambda)f(z)}\right)^{\mu} = u(z), \ z \in \mathbb{E}.$$

Differentiate logarithmically, we obtain:

$$\frac{zI_p'(n+1,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} - \mu\frac{zI_p'(n,\lambda)f(z)}{I_p(n,\lambda)f(z)} + p(\mu-1) = \frac{zu'(z)}{u(z)}. \tag{3}$$

In view of the equality

$$zI_p'(n,\lambda)f(z) = (p+\lambda)I_p(n+1,\lambda)f(z) - \lambda I_p(n,\lambda)f(z)$$

(3) turns to

$$(p+\lambda)\left\{(\mu-1)+\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}-\mu\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right\}=\frac{zu'(z)}{u(z)}. \qquad (4)$$

In view of (2) we get

$$\frac{zu'(z)}{u(z)}\prec\frac{(1-\alpha)z}{1+(1-\alpha)z}=h(z) \quad (say)$$

It is easy to view that $h(z)$ is starlike and $h(0)=0$. Therefore in view of Lemma 1, we conclude that

$$u(z)\prec 1+(1-\alpha)z.$$

Hence

$$\left|\frac{I_p(n+1,\lambda)f(z)}{z}\left(\frac{z}{I_p(n,\lambda)f(z)}\right)^{\mu}-1\right|<1-\alpha, z\in\mathbb{E}.$$

or

$$f\in B_p(n,\lambda,\mu,\alpha).$$

3. Applications to Starlike and Convex functions

Selecting $\lambda=n=0$ in Theorem 2, we obtain:

Corollary 3. *Assume that μ is real number such that $\mu\geq 0$. If $f\in\mathcal{A}_p$ satisfies the condition*

$$1+\frac{zf''(z)}{f'(z)}-\mu\frac{zf'(z)}{f(z)}+p(\mu-1)\prec\frac{(1-\alpha)z}{1+(1-\alpha)z}, \quad 0\leq\alpha<1,$$

then

$$\left|\frac{f'(z)}{p}\left(\frac{z}{f(z)}\right)^{\mu}-1\right|<1-\alpha, \quad z\in\mathbb{E},$$

Putting $\mu=1$ in above corollary, we get:

Corollary 4. *If $f\in\mathcal{A}_p$ satisfies the condition*

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\prec\frac{(1-\alpha)z}{1+(1-\alpha)z}, \quad 0\leq\alpha<1,$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>0, \quad z\in\mathbb{E}, \quad i.e. \quad f\in\mathcal{S}_p^*.$$

Writing $\lambda = 1, n = 0$ in Theorem 2, we obtain:

Corollary 5. *Assume that μ is real number such that $\mu \geq 0$. If $f \in \mathcal{A}_p$ satisfies the condition*

$$\frac{2zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} - \mu \frac{zf'(z)}{f(z)} + p(\mu - 1) \prec \frac{(1-\alpha)z}{1 + (1-\alpha)z}, \quad 0 \leq \alpha < 1,$$

then

$$\left| \frac{f(z) + zf'(z)}{(p+1)z} \left(\frac{z}{f(z)} \right)^{\mu} - 1 \right| < 1 - \alpha, \quad z \in \mathbb{E}.$$

Replacing $\mu = 1$ in above corollary, we get:

Corollary 6. *If $f \in \mathcal{A}_p$ satisfies the condition*

$$\frac{2zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} - \frac{zf'(z)}{f(z)} + \prec \frac{(1-\alpha)z}{1 + (1-\alpha)z}, \quad 0 \leq \alpha < 1,$$

then

$$\left| \frac{1}{p+1} \left(1 + \frac{zf'(z)}{f(z)} \right) - 1 \right| < 1 - \alpha, \quad z \in \mathbb{E}.$$

For $\alpha = 1/2$, the above corollary reduces to the following criterion of starlikeness.

Corollary 7. *If $f \in \mathcal{A}_p$ satisfies*

$$\frac{2zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} - \frac{zf'(z)}{f(z)} + \prec \frac{z}{2+z},$$

then

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > 0, \quad z \in \mathbb{E}, \quad i.e. \quad f \in \mathcal{S}_p^*.$$

Selecting $n = \mu = 1, \lambda = 0$ in Theorem 2, we obtain:

Corollary 8. *If $f \in \mathcal{A}_p$ satisfies the condition*

$$\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + z^2 f''(z)} - \frac{zf''(z)}{f'(z)} \prec \frac{(1-\alpha)z}{1 + (1-\alpha)z}, \quad 0 \leq \alpha < 1,$$

then

$$\left| \frac{1}{p} \left(1 + \frac{zf''(z)}{f'(z)} \right) - 1 \right| < 1 - \alpha, \quad z \in \mathbb{E}.$$

For $\alpha = 0$ in above corollary, we have the following result.

Corollary 9. *If $f \in \mathcal{A}_p$ satisfies the condition*

$$\Re\left(\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + z^2 f''(z)} - \frac{zf''(z)}{f'(z)}\right) < \frac{1}{2}, \quad z \in \mathbb{E},$$

then $f \in \mathcal{K}_p$.

For $p = 1,\ \alpha = 1/2$ in Corollary 8, we obtain the correct version of Corollary 1 of Laura Stanciu and Daniel Breaz [6].

Corollary 10. *If $f \in \mathcal{A}$ satisfies the condition*

$$\Re\left(\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + z^2 f''(z)} - \frac{zf''(z)}{f'(z)}\right) > -1, \quad z \in \mathbb{E},$$

then

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{2}, \quad i.e. \quad f \in \mathcal{K}.$$

Selecting $n = 1, \mu = \lambda = 0$ in Theorem 2, we obtain:

Corollary 11. *If $f \in \mathcal{A}_p$ satisfies the condition*

$$\frac{f'(z) + 3zf''(z) + z^3 f'''(z)}{f'(z) + z^2 f''(z)} \prec p + \frac{(1-\alpha)z}{1 + (1-\alpha)z}, \quad 0 \leq \alpha < 1,$$

then

$$f'(z) + zf''(z) \prec p^2\{1 + (1-\alpha)z\}, \quad z \in \mathbb{E},.$$

For $p = 1,\ \alpha = 1/2$ in Corollary 11, we obtain the correct version of Corollary 2 of Laura Stanciu and Daniel Breaz [6].

Corollary 12. *If $f \in \mathcal{A}$ satisfies the condition*

$$\Re\left(\frac{f'(z) + 3zf''(z) + z^3 f'''(z)}{f'(z) + z^2 f''(z)}\right) > 0, \quad z \in \mathbb{E},$$

then

$$\left|f'(z) + zf''(z) - 1\right| < \frac{1}{2}, \quad z \in \mathbb{E},.$$

Selecting $n = \lambda = 0, \mu = 1$ in Theorem 2, we obtain:

Corollary 13. *If $f \in \mathcal{A}_p$ satisfies the condition*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{(1-\alpha)z}{1+(1-\alpha)z}, \quad 0 \le \alpha < 1,$$

then

$$\frac{zf'(z)}{f(z)} \prec p\{1 + (1-\alpha)z\}, \quad z \in \mathbb{E}.$$

For $p = 1, \ \alpha = 1/2$ in Corollary 13, we obtain the correct version of Corollary 3 of Laura Stanciu and Daniel Breaz [6].

Corollary 14. *If $f \in \mathcal{A}$ satisfies the condition*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{z}{2+z}, \quad z \in \mathbb{E},$$

then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{1}{2}, \quad i.e. \quad f \in \mathcal{S}^*.$$

Selecting $n = \lambda = \mu = 0$ in Theorem 2, we obtain:

Corollary 15. *If $f \in \mathcal{A}_p$ satisfies the condition*

$$1 + \frac{zf''(z)}{f'(z)} \prec p + \frac{(1-\alpha)z}{1+(1-\alpha)z}, \quad 0 \le \alpha < 1,$$

then

$$\left| \frac{f'(z)}{p} - 1 \right| < 1 - \alpha.$$

For $p = 1, \ \alpha = 1/2$ in Corollary 15, we obtain the correct version of Corollary 4 of Laura Stanciu and Daniel Breaz [6].

Corollary 16. *If $f \in \mathcal{A}$ satisfies the condition*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{E},$$

then

$$\left| f'(z) - 1 \right| < \frac{1}{2} \quad i.e. \quad f \in \mathcal{C}.$$

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