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THE NOOR INTEGRAL OPERATOR AND β UNIFORMLY α-SPIRALLIKE FUNCTIONS

E. Amini, Sh. Najafzadeh and A. Ebadian

Abstract. In [10, 12] Noor introduced an integral operator by using convolution. In this paper, we apply this operator on a class of analytic functions. We also apply the proposed operator on β uniformly α-spirallik functions to find some inclusion relations, coefficient bounds and test example.

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1. Introduction

A.W.Goodman investigated about some univalent functions geometrically [3]. For starlik functions, Let Γ_w be the image of an arc $\Gamma_z : z = z(t)\ ; a < t < b$, where $w = f(z)$ and let w_0 be a point not on, Γ_w is starlike with respect to w_0 if $arg(w-w_0)$ is nondecreasing function of t. This condition is equivalent to:

$$Im\Big\{\frac{z'(t)f'(z)}{f(z)-w_0}\Big\} \geq 0.$$

Similarly Γ_w is α-spiral($|\alpha| < \pi/2$) with respect to w_0 if

$$\alpha < arg\Big\{\frac{z'(t)f'(z)}{f(z)-w_0}\Big\} < \alpha + \pi.$$

Let A denot the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the unit disk $\Delta = \{z : |z| < 1\}$.

The class of starlike functions $f \in A$ with respect to origin denote by S^*. If $f \in A$ and f be starlike with respect for every $z \in \Delta$, then f is convex in Δ. The set of all convex functions $f \in A$ denote by $CV.(see[1, 3])$ Similarly the class of α-spirallike functions $f \in A$ with respect to origin denote by $SP(\alpha)$. If $f \in A$ and $zf'(z) \in SP(\alpha)$ then f is convex α-spirallike in Δ. The set of all convex α-spirallike functions $f \in A$ denote by $CVSP(\alpha)$.

For $|\alpha| < \pi/2$, the function $f(z)$ is uniformly α-spirallike if the image of every circular Γ_z with center at ξ lying Δ is α-spirallike with respect to $f(\xi)$. (see [13])

The function $f(z) \in A$ is uniformly α-spirallike in Δ if and only if for every $|\alpha| < \pi/2$, we have

$$Re\Big\{e^{-i\alpha}\frac{(z-\xi)f'(z)}{f(z)-f(\xi)}\Big\} > 0. \quad (see\ [13])$$

For $|\alpha| < \pi/2$ and $0 < \beta < 1$, a function $f(z) \in A$ is said to be β uniformly α-spiral in Δ if for every circular arc Γ_z contained in Δ with center at $\xi(|\xi| < \beta)$ the image of arc $f(\Gamma_z)$ is α-spirallike. (see [17])

The class of all β uniformly α-spirallike function in Δ is denote by $USP(\alpha, \beta)$. (see [17])

Theorem 1. *[17] Let $f \in A$, then $f(z)$ is in $USP(\alpha, \beta)$ if and only if*

$$Re\Big\{e^{-i\alpha}\frac{zf'(z)}{f(z)}\Big\} > \beta\Big|\frac{zf'(z)}{f(z)} - 1\Big|, \qquad z \in \Delta. \tag{2}$$

A function $f(z) \in A$ for all $z \in \Delta$, is said to be in the class of β uniformly convex α-spirallike, written $UCSP(\alpha, \beta)$ if and only if $g(z) = zf'(z)$ and $g(z) \in USP(\alpha, \beta)$. (see [17])

Theorem 2. *[17] Let $f \in A$. $f \in UCSP(\alpha, \beta)$ if and only if,*

$$Re\Big\{e^{-i\alpha}(1 + \frac{zf''(z)}{f'(z)})\Big\} > \beta\Big|\frac{zf''(z)}{f'(z)}\Big|, \qquad z \in \Delta. \tag{3}$$

Let $f(z)$ and $g(z)$ be analytic in Δ. Then $f(z)$ is said to be subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1 (z \in \Delta)$ such that $f(z) = g(w(z))$ for $z \in \Delta$.

$g(z)$ is univalent in Δ, $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$. (see [7])

Theorem 3. *[17] Let $f \in A$, $0 < \beta < 1$, then the function f(z) is in $USP(\alpha, \beta)$ if and only if,*

$$e^{-i\alpha}\frac{zf'(z)}{f(z)} \prec h_\beta(z)cos\alpha - isin\alpha,$$

where

$$h_\beta(z) = 1 + \frac{1}{2sin^2(\sigma)}\Big\{\Big(\frac{1+\sqrt{x}}{1-\sqrt{z}}\Big)^{\frac{2\sigma}{\pi}} + \Big(\frac{1-\sqrt{z}}{1+\sqrt{z}}\Big)^{\frac{2\sigma}{\pi}} - 2\Big\}. \tag{4}$$

and $\sigma = arccos\beta$.

Note that $h_\beta(0) = 1$ and h_β maps Δ conformally onto the hyperbolic regain

$$D_\beta = \big\{w = u + iv : u > \beta\sqrt{(u-1)^2 + v^2}\big\}.$$

Since D_β is a convex regoin, h_β is convex (and univalent) in Δ. (see [4, 17])

K.I. Noor and M.A. Noor defined an integral operator $I_n : A \longrightarrow A$ as follows.

$$I_n f(z) = f_n^\dagger(z) * f(z). \tag{5}$$

where $f_n^\dagger$ is defined by the relation

$$\frac{z}{(1-z)^{n+1}} * f_n^\dagger(z) = \frac{z}{(1-z)^2}. \quad (see[10, 12]) \tag{6}$$

It is obvious that $I_0(z) = zf'(z)$ and $I_1(z) = f(z)$. The operator $I_n f$ defined by (5) is called the Noor integral operator of nth order of f.

J.L. Liu prove that the Noor integral operator satisfying the equation

$$z(I_{n+1}f(z))' = (n+1)I_n f(z) - nI_{n+1}f(z). \quad (see[5]) \tag{7}$$

Liu and Noor [6] investigated some interesting properties of the Noor integral operator and applications of the Noor integral operator. (for more details see [9, 11])

It is well known that for $\alpha > 0$

$$\frac{z}{(1-z)^\alpha} = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{m!} z^{m+1}, \qquad (z \in \Delta).$$

where $(\alpha)_m$ is the Pochhammer symbol

$$(\alpha)_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)} = \begin{cases} 1, & n = 0,\, \alpha \neq 0 \\ \alpha(\alpha+1)...(\alpha+m-1), & n \in \mathbb{N}. \end{cases}$$

By (6) we obtain,

$$\sum_{m=0}^{\infty} \frac{(n+1)_m}{m!} z^{m+1} * f_n^\dagger(z) = \sum_{m=0}^{\infty} \frac{(2)_m}{m!} z^{m+1}. \tag{8}$$

Then (8) implies that

$$f_n^{\dagger}(z) = \sum_{m=0}^{\infty} \frac{(2)_m}{(n+1)_m} z^{m+1}, \qquad (z \in \Delta).$$

Therefor, if f is of the form (1), then

$$I_n f(z) = z + \sum_{m=2}^{\infty} \frac{(2)_{m-1}}{(n+1)_{m-1}} a_m z^m = z + \sum_{m=2}^{\infty} \frac{m!}{(n+1)_{m-1}} a_m z^m, \quad (z \in \Delta).$$

In the present paper we give some argument properties of β uniformly α-spirallike functions and investigate some properties of the Noor integral operator.

2. Preliminary lemmas

We need the following Lemmas for our investigation.

Lemma 4. *[15] Let $0 < \alpha < \beta$. If $\beta \geq 2$ or $\alpha + \beta \geq 3$, then the function*

$$h(z) = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\beta)_m} z^{m+1}, \qquad (z \in \Delta).$$

belongs to the class of convex functions.

Lemma 5. *[15, 16] If $f \in CV$ and $g \in SP(\alpha)$, then for each analytic function h in Δ with $h(0) = 1$,*

$$\frac{(\tilde{f} * hg)(\Delta)}{(\tilde{f} * g)(\Delta)} \subseteq \bar{co}h(\Delta),$$

where $\tilde{f}(z) = f(\frac{z}{2})$ and $\bar{co}h(\Delta)$ denotes the closed convex hull of $h(\Delta)$.

Lemma 6. *[2, 8] Let f be convex univalent in Δ with $f(0) = 1$ and $Re(\lambda f(z) + \mu) > 0$ $(\lambda, \mu \in \mathbb{C})$. If p is analytic in Δ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\lambda p(z) + \mu} \prec f(z).$$

implies,

$$p(z) \prec f(z), \qquad (z \in \Delta).$$

Remark 1. *Let $|\alpha| < \pi/2$ and f be a convex univalent function in Δ with $g(0) = e^{-i\alpha}$ and $Re(\lambda g(z) + \mu) > 0$ $(\lambda, \mu \in \mathbb{C})$. If p is analytic in Δ with $p(0) = e^{-i\alpha}$ then*

$$p(z) + \frac{zp'(z)}{\lambda p(z) + \mu} \prec g(z).$$

implies,

$$p(z) \prec g(z), \qquad (z \in \Delta).$$

Lemma 7. *[14] Let*

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \prec 1 + \sum_{n=1}^{\infty} C_n z^n = H(z), \quad (z \in \Delta).$$

If the function H be univalent in Δ and $H(\Delta)$ be a convex set, then

$$|c_n| \leq |C_1|.$$

Lemma 8. *[17] If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in SP(\alpha, \beta)$ and $0 < \beta < 1$, then*

$$|a_2| \leq 8cos\alpha\Big(\frac{\sigma}{\pi sin\sigma}\Big)^2, \qquad \sigma = arccos\beta. \tag{9}$$

This result is sharp. Also it is clear that

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + ..., \qquad (z \in \Delta).$$

3. Main results

Theorem 9. *Let $f \in A$. If $f \in SP(\alpha, \beta)$ satisfying the condition*

$$\frac{e^{-i\alpha} z(I_n f(z))'}{I_n f(z)} \prec h_\beta(z)cos\alpha - isin\alpha, \quad (z \in \Delta), \tag{10}$$

then,

$$\frac{e^{-i\alpha} z(I_{n+1} f(z))'}{I_{n+1} f(z)} \prec h_\beta(z)cos\alpha - isin\alpha, \quad (z \in \Delta). \tag{11}$$

Proof. Let

$$p(z) = \frac{e^{-i\alpha} z(I_{n+1} f(z))'}{I_{n+1} f(z)},$$

where p is an analytic function with $p(0) = e^{-i\alpha}$. By using the equation (7) , we have

$$p(z) + e^{-i\alpha}n = e^{-i\alpha}(n+1)\frac{I_n f(z)}{I_{n+1}f(z)}. \tag{12}$$

Taking logarithmic derivative in both side of (12) and multiplying by $e^{-i\alpha}z$, we have

$$p(z) + \frac{zp'(z)}{e^{i\alpha}p(z)+n} = \frac{e^{-i\alpha}(I_n f(z))'}{I_n f(z)}.$$

By applying relation (10) and Remark 1 it follows that $p(z) \prec h_\beta(z)cos\alpha - isin\alpha$ that is the relation (11).

Theorem 10. *If a function $f \in A$ satisfies the condition*

$$\frac{e^{-i\alpha}z\Big(I_n f(z)\Big)'}{I_n f(z)} \prec h_\beta(z)cos\alpha - isin\alpha, \qquad (z \in \Delta), \tag{13}$$

then,

$$\frac{e^{-i\alpha}z\Big(I_n F_c(f)(z)\Big)'}{I_n F_c(f)(z)} \prec h_\beta(z)cos\alpha - isin\alpha, \qquad (z \in \Delta), \tag{14}$$

where F_c be the integral operator defined by

$$F_c(f)(z) = \frac{c+1}{c}\int_0^z t^{c-1}f(t)dt, \quad (c \geq 0). \tag{15}$$

Proof. Let

$$p(z) = \frac{e^{-i\alpha}z\Big(I_n F_c(f)(z)\Big)'}{I_n F_c(f)(z)},$$

where p is analytic function with $p(0) = e^{-i\alpha}$. From (15) we have

$$z\Big(I_n F_c(f)\Big)'(z) = (c+1)I_n f(z) - cI_n F_c(f)(z). \tag{16}$$

Then by using (16) ,we get

$$c + p(z) = (c+1)\frac{e^{-i\alpha}I_n f(z)}{I_n F_c(f)(z)}. \tag{17}$$

Taking logaritmic derivatives in both side of (17) and multiplying by $e^{-i\alpha}z$, we have

$$p(z) + \frac{zp'(z)}{e^{i\alpha}c + e^{i\alpha}p(z)} = \frac{e^{-i\alpha}z(I_n f(z))'}{I_n f(z)}. \tag{18}$$

Therefor by relations (13) and (18) and Remark 1 we obtain (14) for all $z \in \Delta$ and the proof is complete.

Definition 1. *A function $f \in A$ is said to be in the class $M_\alpha(n,\beta)$ ($|\alpha| < \pi/2,\ 0 \le \beta \le 1$) if and only if, $I_n(f) \in USP(\alpha,\beta)$ or equivalently*

$$Re\Big\{\frac{e^{-i\alpha} z(I_n f)'(z)}{(I_n f)(z)}\Big\} > \beta\Big|\frac{z(I_n f)'(z)}{(I_n f)(z)} - 1\Big|, \qquad (z \in \Delta). \tag{19}$$

Note that the class $M_\alpha(n,\beta)$ unifies many subclasses of A. In particular, $M_\alpha(1,0) = CVSP(\alpha)$, the class of convex α-spirallike functions; $M_\alpha(0,0) = SP(\alpha)$, the class of α-spirallike functions; $M_\alpha(1,1) = USP(\alpha)$, the class of uniformly α-spirallike functions; $M_\alpha(0,1) = UCSP(\alpha)$, the class of uniformly convex α-spirallike functions; $M_\alpha(0,\beta) = UCSP(\alpha,\beta)$ and $M_\alpha(1,\beta) = USP(\alpha,\beta)$.

Also, by a simple computation, if $0 < \beta_1 < \beta_2 < 1$ then $M_\alpha(n,\beta_2) \subset M_\alpha(n,\beta_1)$.

Theorem 11. *The function $k(z) = \frac{z}{(1-Az)^{1+i}}$ is in $M_\alpha(1,\beta)$ if and only if*

$$|A| \le \frac{cos\alpha}{\beta\sqrt{2} + sin\alpha}. \tag{20}$$

Proof. By using (2) $k(z) \in USP(\alpha,\beta)$, if and only if

$$Re\Big\{e^{-i\alpha}\frac{1+Aiz}{1-Az}\Big\} \ge \beta\Big|\frac{Az(i+1)}{1-Az}\Big|. \tag{21}$$

It is suffices to consider $|z| = 1$ in the above relation, by setting $|A| = r$ and $Az = re^{i\theta}$ we have $Aiz = re^{i(\theta+\frac{\pi}{2})}$. It follows from (21),

$$Re\Big\{e^{-i\alpha}\frac{1+re^{i(\theta+\frac{\pi}{2})}}{1-re^{i\theta}}\Big\} \ge \frac{\beta r\sqrt{2}}{|1-re^{i\theta}|}. \tag{22}$$

After simplification, we see that

$$Re\Big\{e^{-i\alpha}\frac{1+re^{i(\theta+\frac{\pi}{2})}}{1-re^{i\theta}}\Big\} = \frac{cos\alpha(1-rcos\theta-rsin\theta)+rsin\alpha(sin\theta+cos\theta-r)}{|1-re^{i\theta}|^2}. \tag{23}$$

By using (22), (23), it is equivalent to

$$\frac{cos\alpha(1-rcos\theta-rsin\theta)+rsin\alpha(sin\theta+cos\theta-r)}{(1-2rcos\theta+r^2)^{\frac{1}{2}}} \ge \beta r\sqrt{2}. \tag{24}$$

The minimum value of the expression in the left hand side of the equation (24) occur at $\theta = \pi$ and this minimum value is $cos\alpha - rsin\alpha$, so we have

$$r \le \frac{cos\alpha}{\beta\sqrt{2} + sin\alpha}. \tag{25}$$

Hence, a necessary and sufficient condition for (20) is (25).

Example 1. *The function $\varphi(z) = z + a_m z^m \in UCSP(\alpha, \beta)$ if and only if it satisfies (3). It is suffices to consider $|z| = 1$ in the above relation, by setting $|ma_m| = r$ and $ma_m z^{m-1} = re^{i\theta}$, we have*

$$Re\Big\{e^{-i\alpha}\frac{1 + mre^{i\theta}}{1 + re^{i\theta}}\Big\} \geq \frac{\beta(m-1)r}{|1 + re^{i\theta}|}. \tag{26}$$

After simplifying and separating the real part of the expression of (26), we get

$$\frac{cos\alpha(1 + mr^2 + mrcos\theta + rcos\theta) - rsin\alpha sin\theta(m-1)}{\{1 + r^2 + 2rcos\theta\}^{\frac{1}{2}}} \geq \beta(m-1)r.$$

The minimum of the expression in the left hand side of the above equation occurs at $\theta = \pi$ and this minimum value is $cos\alpha(1 - mr)$, hence

$$r \leq \frac{cos\alpha}{(m-1)\beta + mcos\alpha}.$$

After by solving this equation for $r = |ma_m|$, we have

$$|a_m| \leq \frac{cos\alpha}{m(m-1)\beta + m^2 cos\alpha}.$$

Since the function $f(z) \in A$ is β uniformly convex α-spirallike in Δ if and only if $zf'(z)$ is β uniformly α-spirallike in Δ, yields; if $f \in USP(\alpha, \beta)$ then,

$$|a_m| \leq \frac{cos\alpha}{(m-1)\beta + mcos\alpha}.$$

If $\varphi \in USP(\alpha, \beta)$, then

$$I_n\varphi(z) = z + \frac{m!}{(n+1)_{m-1}} a_m z^m,$$

is in $UCSP(\alpha, \beta)$ for $n \in \{3, 4, ...\}$. Moreover $I_n\varphi \notin UCSP(\alpha, \beta)$ for $n \in \{1, 2\}$. It would be interesting to check this property of the Noor integral operator for other functions in $USP(\alpha, \beta)$.

Theorem 12. *The function $f(z) = z + a_m z^m$ is in $M_\alpha(n, \beta)$ if and only if*

$$|a_m| \leq \frac{(n+1)_{m-1} cos\alpha}{m!\big((m-1)\beta + mcos\alpha\big)}, \quad (m \geq 2).$$

Proof. Let $I_nf(z) = z + b_m z^m = z + \frac{m!}{(n+1)_{m-1}} a_m z^m$. It is suffices to consider $|z| = 1$ in the above relation, by setting $|b_m| = r$ and $b_m z^{m-1} = re^{i\theta}$, then (19) for this f will be

$$Re\Big\{e^{-i\alpha}\frac{1+mre^{i\theta}}{1+re^{i\theta}}\Big\} \geq \frac{\beta r(m-1)}{|1+re^{i\theta}|}.$$

By the same steps of theorem 11, we get the desired result.

Remark 2. *For particular value of m, n, β, Theorem 12 provides functions belonging to the class $M_\alpha(n,\beta)$. For example for $m = 2, n = 0, \beta = 1$, we have*

$$|a_2| \leq \frac{cos\alpha}{2+4cos\alpha},$$

so the function $f(z) = z + \frac{cos\alpha}{2+4cos\alpha}z^2$ is in $UCSP(\alpha)$.

Theorem 13. *Let f with the form (1), be in the class $M_\alpha(n,\beta)$, then*

$$|a_2| \leq \frac{n+1}{2}cos\alpha\big(\frac{\sigma}{\pi sin\sigma}\big)^2, \qquad (\sigma = arccos\beta), \tag{27}$$

and

$$|a_m| \leq \frac{(n+1)_{m-1}}{(m-1)(2)_{m-1}}cos\alpha\Big(\frac{\sigma}{\pi sin\sigma}\Big)^2\prod_{t=3}^{m}\Big(1+\frac{cos\alpha}{t-2}\Big(\frac{\sigma}{\pi sin\sigma}\Big)^2\Big), \ (\sigma = arccos\beta). \tag{28}$$

Proof. Let f is given by (1), belongs to $M_\alpha(n,\beta)$, also $I_nf(z) = z + \sum_{m=2}^{\infty} b_m z^m = F(z)$, where

$$b_m = \frac{(2)_{m-1}}{(n+1)_{m-1}}a_m. \tag{29}$$

We define

$$\varphi(z) = e^{-i\alpha}\frac{zF'(z)}{F(z)} = e^{-i\alpha} + \sum_{m=1}^{\infty} c_m z^m.$$

Then by using theorem 3, we have $e^{i\alpha}\varphi(z) \prec e^{i\alpha}(cos\alpha h_\beta z - isin\alpha)$, where h_β is given by (4) depending on β and the function h_β is univalent in Δ and $h_\beta(\Delta) = D_\beta$.

Using Rogosinski lemma 7 and relation (9) of lemma 8 for function $e^{-i\alpha}\varphi(z)$, we have $|e^{-i\alpha}c_m| \leq |a_2|$. Now, writing $e^{-i\alpha}\varphi(z)F(z) = zF'(z)$ and comparing the coefficients of z^n on both sides, we get

$$(m-1)b_m = \sum_{k=1}^{m-1} e^{i\alpha}c_{m-k}b_k.$$

Form the above equality, we get $|b_2| = |c_1| \leq |a_2|$. By using the equation (29) we obtain (28).

Further

$$|b_3| = \frac{1}{2}|e^{i\alpha}c_2 + e^{i\alpha}c_1b_2| \leq \frac{1}{2}(|c_1| + |c_2|)b \leq \frac{1}{2}a_2(1+a_2).$$

By using the induction, we have

$$|b_k| \leq \frac{a_2}{k-1}(1+a_2)(1+\frac{a_2}{2})...(1+\frac{a_2}{k-2}), \quad k = 3,4,...,m-1.$$

Then,

$$\begin{aligned}(m-1)|b_m| &\leq \sum_{k=1}^{m-1} |c_{m-k}||b_k| \leq a_2 \sum_{k=1}^{m-1} |b_k| \\ &\leq a_2\Big(1 + a_2 + \frac{a_2}{2}(1+a_2) + \frac{a_2}{3}(1+a_2)(1+\frac{a_2}{2}) + ... \\ &+ \frac{a_2}{m-2}(1+a_2)(1+\frac{a_2}{2})...(1+\frac{a_2}{m-3})\Big) \\ &= a_2(1+a_2)(1+\frac{a_2}{2})...(1+\frac{a_2}{m-2}),\end{aligned}$$

and hence

$$|b_m| \leq \frac{a_2}{m-1}\prod_{t=3}^{m}\big(1+\frac{a_2}{t-2}\big), \quad (m \geq 3).$$

By using (29) and (9) we obtain (27).

Theorem 14. *Assume that* $n_1 \leq n_2, n_1, n_2 \in \mathbb{N}\bigcup\{0\}$, *then*

$$M_\alpha(n_1, k) \subseteq M_\alpha(n_2, k),$$

for all $k \in (0, \infty)$ *and* $|z| < 1/2$.

Proof. Let $f \in M_\alpha(n,k)$. By definition 1 and theorem 3 we have

$$\frac{z(I_{n_1 f(z)})'}{I_{n_1}f(z)} = h_\beta(w(z))cos\alpha - isin\alpha, \tag{30}$$

where $h_\beta(\Delta) = D_\beta$ and $|w(z)| < 1$ in Δ with $w(0) = 0$.

Let us denote

$$f_{n_1,n_2} = \sum_{m=0}^{\infty}\frac{(n_1+1)}{(n_2+1)}z^{m+1}, \qquad (z \in \Delta). \tag{31}$$

Then by (31), we have

$$f^{\dagger}_{n_2}(z) = f^{\dagger}_{n_1}(z) * f^{\dagger}_{n_1,n_2}(z).$$

Applying (5), (30), (31) and the properties of convolution, we get

$$\begin{aligned} e^{-i\alpha} z\frac{(I_{n_2}f)'(\frac{z}{2})}{I_{n_2}f(\frac{z}{2})} &= e^{-i\alpha} z\frac{(f^{\dagger}_{n_2} * f)'(\frac{z}{2})}{(f^{\dagger}_{n_2} * f)(\frac{z}{2})} \\ &= e^{-i\alpha} z\frac{(f^{\dagger}_{n_1} * f_{n_1,n_2} * f)'(\frac{z}{2})}{(f^{\dagger}_{n_1} * f_{n_1,n_2} * f)(\frac{z}{2})} \\ &= e^{-i\alpha}\frac{f_{n_1,n_2}(\frac{z}{2}) * z(I_{n_1}f(z))'}{f_{n_1,n_2}(\frac{z}{2}) * I_{n_1}f(z)} \\ &= \frac{f_{n_1,n_2}(\frac{z}{2}) * \big(h_\beta(w(z))cos\alpha - isin\alpha\big)I_{n_1}f(z)}{f_{n_1,n_2}(\frac{z}{2}) * I_{n_1}f(z)}. \end{aligned} \tag{32}$$

Moreover, it follows from (30) that $I_{n_1}f \in USP(\alpha) \subseteq SP(\alpha) \subseteq S^*$ and obtain from lemma 4, $f_{n_1,n_2} \in CV$. Then by using lemma 5 to (32), we obtain

$$\frac{f_{n_1,n_2}(\frac{z}{2}) * \big(h_\beta(w(z))cos\alpha - isin\alpha\big)I_{n_1}f(z)}{f_{n_1,n_2}(\frac{z}{2}) * I_{n_1}f(z)} \subseteq \bar{co}\big(h_\beta(w(z))cos\alpha - isin\alpha\big), \quad (z \in \Delta).$$

Hence the function (32) is subordinated to $h_\beta(z)cos\alpha - isin\alpha$, so $f \in M_\alpha(n_2, \beta)$ for $|z| < 1/2$.

Corollary 15. *The Theorem* 14 *are satisfied*

$$USP(\alpha, \beta) = M_\alpha(1, \beta) \subset M_\alpha(n, \beta),$$

for all $|z| < 1/2$, $\beta \in (0, 1)$ *and all* $n \in \mathbb{N}$.

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Ebrahim Amini
Department of Mathematics, Faculty of Science,
Payame Noor University,
P.O. Box 19395-3697 Tehran, Iran
email: *ab.amini.s@gmail.com*

Shahram Najafzadeh
Department of Mathematics, Faculty of Science,
Payame Noor University,

P.O. Box 19395-3697 Tehran, Iran
email: *najafzadeh1234@yahoo.ie*

Ali Ebadian
Department of Mathematics, Faculty of Science,
Payame Noor University,
P.O. Box 19395-3697 Tehran, Iran
email: *ebadian.ali@gmail.com*