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ON COEFFICIENT DETERMINANTS INVOLVING MANY FEKETE-SZEGO-TYPE PARAMETERS OF CONVEX FUNCTIONS

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Abstract. We examine the Hankel determinants involving many Fekete-Szego-type parameters for the class of convex functions of analytic mappings of the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, which satisfy the condition

$$Re \left[1 + \frac{zf''(z)}{f'(z)}\right] > 0, \ z \in U.$$

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1. Introduction

A classical problem settled by Fekete and Szego [6] is to find the maximum value of the coefficient functional $|a_3 - \lambda a_2^2|$ for each λ over the class A of univalent functions f in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in U \tag{1.1}$$

In [11], Noonan and Thomas defined the qth Hankel determinants of f for $q \geq 1$, $n \geq 0$ by:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

This determinant has been studied for specific choices of q and n by several authors and include the Fekete-Szego functional as a special case for $\lambda = 1, q = 2$ and $n = 1$. The definitions given by Babalola [3], called Hankel determinants with Fekete-Szego parameter as:

Definition 1. *Let λ be a nonnegative real number. Then for integers $n \geq 1$ and $q \geq 1$, we define the qth-Hankel determinants with Fekete-Szego parameter λ, $H_q^\lambda(n)$ as:*

$$H_q^\lambda(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & \lambda a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

Definition 2. *Let λ be a nonnegative real number. Then for integers $n \geq 1$ and $q \geq 1$, we define the $B_q^\lambda(n)$ determinants as:*

$$B_q^\lambda(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & \lambda a_{n+2(q-1)} \end{vmatrix}$$

have been extended to include many Fekete-Szego-type parameters in Babalola [4] to accomodate a wide variety of emerging functionals in the study of coefficients of mappings of the unit disk and are given by

Definition 3. *Let λ_i, $i = 1, 2, ..., q$ be nonnegative real numbers. Then for integers $n \geq 1$ and $q \geq 1$, the qth-Hankel determinants with many Fekete-szego- type parameter λ_i, $H_q^{\lambda_1,\lambda_2,...,\lambda_q}(n)$ is define as:*

$$H_q^{\lambda_1,\lambda_2,...,\lambda_q}(n) = \begin{vmatrix} \lambda_1 a_n & \lambda_2 a_{n+1} & \cdots & \lambda q a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

and

Definition 4. *Let $\lambda_j = 1, 2, ..., n$ be nonnegative real numbers. Then for integers $n \geq 1$ and $q \geq 1$, $B_q^{\lambda_1,\lambda_2,...,\lambda_n}(n)$ determinants is given by:*

$$B_q^{\lambda_1,\lambda_2,...\lambda_n}(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & \lambda_1 a_{n+q-1} \\ a_{n+q} & a_{n+q+1} & \cdots & \lambda_2 a_{n+2q-1} \\ a_{n+2q} & a_{n+2q+1} & \cdots & \lambda_3 a_{n+3q-1} \\ \vdots & \cdots & \cdots & \vdots \\ a_{n+q(q-1)} & \cdots & \cdots & \lambda_n a_{n+q^2-1} \end{vmatrix}$$

for $\lambda_j = 1,\ j = 1, 2, ..., q-1 (n-1)$, we write $H_q^{\lambda}(n)$ and $B_q^{\lambda}(n)$ in place of $H_q^{1,1,...,1,\lambda_q}(n)$ and $B_q^{1,1,...,1,\lambda_n}(n)$ respectively. Note that for real numbers γ, α and β, we obtain

$$H_2^{\gamma}(1) = \begin{vmatrix} 1 & \gamma a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - \gamma a_2^2$$

$$H_2^{\alpha}(2) = \begin{vmatrix} a_2 & \alpha a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - \alpha a_3^2$$

and

$$B_2^{\beta}(1) = \begin{vmatrix} 1 & a_2 \\ a_3 & \beta a_4 \end{vmatrix} = a_2 a_3 - \beta a_4$$

In this work, we shall examine the determinant $H_3^{\lambda_1,\lambda_2,\lambda_3}(1)$ for the class of convex functions satisfying the geometric condition

$$Re\ \left[1 + \frac{zf''(z)}{f'(z)}\right] > 0,\ z \in U.$$

denoted by C. By definition

$$H_3^{\lambda_1,\lambda_2,\lambda_3}(1) = \begin{vmatrix} \lambda_1 a_1 & \lambda_2 a_2 & \lambda_3 a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

for $f \in C$, $a_1 = 1$ so that

$$\begin{aligned} H_3^{\lambda_1,\lambda_2,\lambda_3}(1) &= a_3(\lambda_2 a_2 a_4 - \lambda_3 a_3^2) - a_4(\lambda_1 a_4 - \lambda_3 a_2 a_3) + a_5(\lambda_1 a_3 - \lambda_2 a_2^2) \\ &= \lambda_2 a_3(a_2 a_4 - \alpha a_3^2) + \lambda_3 a_4(a_2 a_3 - \beta a_4) + \lambda_1 a_5(a_3 - \gamma a_2^2) \end{aligned}$$

where $\alpha = \frac{\lambda_3}{\lambda_2}$, $\beta = \frac{\lambda_1}{\lambda_3}$ and $\gamma = \frac{\lambda_2}{\lambda_1}$. By triangle inequality, we have

$$|H_3^{\lambda_1,\lambda_2,\lambda_3}(1)| \leq \lambda_1|a_5||H_2^{\gamma}(1)| + \lambda_2|a_3||H_2^{\alpha}(2)| + \lambda_3|a_4||B_2^{\beta}(1)| \quad (1.2)$$

For the class C of convex function, the Fekete-Szego functional, $|H_2^{\gamma}(1)|$ is known and given by:

Theorem 1. *[8] If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic and convex in U and if γ is a complex number, then*

$$|a_3 - \gamma a_2^2| = \max\left\{\frac{1}{3}, |\gamma - 1|\right\}$$

The result is sharp for each γ.

We shall examine the best possible bounds on $H_2^{\alpha}(2)$ and $B_2^{\beta}(1)$ to conclude our investigation.

2. Preliminary lemmas

To prove the main results in the next section, we need the following lemmas. Let P denote the class of Caratheodory functions $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ which are analytic and satisfy $p(0) = 1$, Re $p(z) > 0$ in open unit disk U.

Lemma 2. *[5] Let $p \in P$. Then $|c_k| \leq 2, k = 1, 2, 3....$ Equality is attained by the moebius function*

$$L_0(z) = \frac{1+z}{1-z}.$$

Lemma 3. *[1] Let $p \in P$. Then we have sharp inequalities*

$$\left| c_2 - \sigma \frac{c_1^2}{2} \right| \leq \begin{cases} 2(1-\sigma), & \text{if } \sigma \leq 0, \\ 2, & \text{if } 0 \leq \sigma \leq 2, \\ 2(\sigma - 1), & \text{if } \sigma \geq 2. \end{cases}$$

For each σ, equality is attained by p(z)given by

$$p(z) = \begin{cases} \frac{1+z^2}{1-z^2}, & \text{if } 0 \leq \sigma \leq 2, \\ \frac{1+z}{1-z}, & \text{if } \sigma \in [-\infty, 0] \cup [2, \infty] \end{cases}$$

Lemma 4. *[5] If f is convex, then $|a_k| \leq 1$. Equality is attained by the function $f(z) = \frac{1}{1-z}$*

Lemma 5. *[9, 10] Let $p \in P$, then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.1}$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{2.2}$$

for some value of x,z such that $|x| \leq 1$ and $|z| \leq 1$.

3. Main results

Theorem 6. *Let $f \in C$, then for real number β,*

$$|B_2^{\beta}(1)| \leq \max\left\{\frac{\beta}{6}, |\beta - 1|\right\}$$

Proof. If $f \in C$, then there exist a $p \in P$ such that

$$1 + \frac{zf''(z)}{f'(z)} = p(z).$$

Equating coefficients of $f'(z)+zf''(z)$ and $f'(z)p(z)$, we find that $a_2 = \frac{c_1}{2}$, $a_3 = \frac{c_1^2+c_2}{6}$ and $a_4 = \frac{2c_3+3c_1c_2+c_1^3}{24}$. Therefore

$$|B_2^{\beta}(1)| = |a_2a_3 - \beta a_4| = \left|\frac{(2-3\beta)}{24}c_1c_2 + \frac{(2-\beta)}{24}c_1^3 - \frac{\beta}{12}c_3\right| \tag{3.1}$$

we observe that if $\beta \leq \frac{2}{3}$, then (3.1) gives

$$|B_2^{\beta}(1)| \leq \left|\frac{(2-\beta)}{24}c_1^3 - \frac{\beta}{12}c_3\right| + \left|\frac{(2-3\beta)}{24}c_1c_2\right|$$

applying lemma (2), we obtain

$$|B_2^{\beta}(1)| \leq \left|\frac{(2-\beta)}{24}c_1^3 - \frac{\beta}{12}c_3\right| + \frac{2-3\beta}{6} \tag{3.2}$$

substituting for c_3 using lemma(5), we have

$$\left|\frac{(2-\beta)}{24}c_1^3 - \frac{\beta}{12}c_3\right| = \left|\frac{(4-3\beta)}{48}c_1^3 - \frac{\beta c_1(4-c_1^2)x}{24} + \frac{\beta c_1(4-c_1^2)x^2}{48} - \frac{\beta(1-|x|^2)(4-c_1^2)z}{24}\right|$$

By lemma (2), $|c_1| \leq 2$. Letting $c_1 = c$, we may assume without restriction that $c \in [0,2]$. Applying the triangle inequality with $\rho = |x|$, we get

$$\left|\frac{(2-\beta)}{24}c_1^3 - \frac{\beta}{12}c_3\right| \leq \frac{(4-3\beta)}{48}c^3 + \frac{\beta(4-c^2)}{24} + \frac{\beta c(4-c^2)\rho}{24} + \frac{\beta(c-2)(4-c^2)\rho^2}{48} = F(\rho)$$

The extreme values of $F(\rho)$ are at $\rho = 0, \rho = 1$ and ρ such that

$$F'(\rho) = \frac{\beta c(4-c^2)}{24} + \frac{\beta(c-2)(4-c^2)\rho}{24} = 0$$

which implies $\rho = \frac{c}{2-c}$.
Let

$$G_1(c) = F(0) = \frac{(4-3\beta)}{48}c^3 + \frac{\beta(4-c^2)}{24}$$

$$G_2(c) = F(1) = \frac{(2-3\beta)}{24}c^3 + \frac{\beta}{4}c$$

and

$$G_3(c) = F(c/2-c) = \frac{(2-\beta)c^3}{24} + \frac{\beta}{6}$$

By elementary calculus, we find that $G_1(c) \le G_1(0) = \frac{\beta}{6}$, $G_2(c) \le G_2(2) = \frac{4-3\beta}{6}$ and $G_3(c) \le G_3(0) = \frac{\beta}{6}$. Hence for $\beta \le \frac{2}{3}$, the maximum of the functional is $\frac{4-3\beta}{6}$. Using this in (3.2), we obtain

$$|B_2^\beta(1)| \le \frac{4-3\beta}{6} + \frac{2-3\beta}{6} \le 1-\beta$$

Next, we consider the case $\frac{2}{3} < \beta \le 2$. Then we write (3.1) as

$$|B_2^\beta(1)| = \left|\frac{\beta}{12}c_3 + \frac{(3\beta-2)}{24}c_1c_2 - \frac{(\beta-2)}{24}c_1^3\right|$$

substituting for c_3 and c_2 by lemma (5)

$$\begin{aligned} |B_2^\beta(1)| = \Bigg| &\frac{(\beta+1)c_1^3}{24} + \frac{(5\beta-2)c_1(4-c_1^2)x}{48} - \frac{\beta c_1(4-c_1^2)x^2}{48} \\ &+ \frac{\beta(1-|x|^2)(4-c_1^2)z}{24}\Bigg| \end{aligned} \tag{3.3}$$

then we can write (3.3) as

$$\begin{aligned} |B_2^\beta(1)| = \Bigg| &\frac{-(\beta+1)c_1^3}{24} - \frac{(5\beta-2)c_1(4-c_1^2)x}{48} + \frac{\beta c_1(4-c_1^2)x^2}{48} \\ &- \frac{\beta(1-|x|^2)(4-c_1^2)z}{24}\Bigg| \end{aligned}$$

Also by lemma (2), $|c| \le 2$. Then letting $c_1 = c$ we may assume without restriction that $c \in [0, 2]$. Applying triangle inequality with $|x| = \rho$, we obtain

$$\begin{aligned} |B_2^\beta(1)| &\le \frac{(\beta+1)c^3}{24} + \frac{\beta(4-c^2)}{24} + \frac{(5\beta-2)c(4-c^2)\rho}{48} + \frac{\beta(c-2)(4-c^2)\rho^2}{48} \\ &= F(\rho) \end{aligned}$$

$$F'(\rho) = \frac{(5\beta-2)c(4-c^2)}{48} + \frac{\beta(c-2)(4-c^2)\rho}{24}$$

which is negative whenever $\frac{2}{3} < \beta \leq 2$. Hence $F(\rho)$ is decreasing on [0,1], so that $F(\rho) \leq F(0)$. That is

$$\begin{aligned} F(\rho) &\leq \frac{(\beta+1)c^3}{24} + \frac{\beta(4-c^2)}{24} \\ &= G(c) \end{aligned}$$

Hence $G(c) \leq G(0) = \frac{\beta}{6}$. Therefore $|B_2^{\beta}(1)| \leq \frac{\beta}{6}$ for $\frac{2}{3} < \beta \leq 2$.
Finally, if $\beta > 2$, we write (3.1) as

$$|B_2^{\beta}(1)| = \left| \frac{\beta}{12}c_3 + \frac{(3\beta-2)}{24}c_1c_2 + \frac{(\beta-2)}{24}c_1^3 \right|$$

and applying lemma (2), we get

$$|B_2^{\beta}(1)| \leq \beta - 1$$

hence

$$\left|B_2^{\beta}(1)\right| \leq \begin{cases} 1-\beta, & \text{if } 0 \leq \beta \leq \frac{2}{3}, \\ \frac{\beta}{6}, & \text{if } \frac{2}{3} < \beta \leq 2, \\ \beta - 1, & \text{if } \beta > 2. \end{cases}$$

Corollary 7. *[2] Let $f \in C$. Then*

$$|B_2(1)| \leq \frac{1}{6}$$

Theorem 8. *Let $f \in C$. Then for real number α*

$$|H_2^{\alpha}(2)| \leq \max\left\{\frac{1}{8}, |\alpha - 1|\right\}$$

Proof. If $f \in C$, then as in above, $a_2 = \frac{c_1}{2}$, $a_3 = \frac{c_1^2+c_2}{6}$ and $a_4 = \frac{2c_3+3c_1c_2+c_1^3}{24}$, so that

$$|H_2^{\alpha}(2)| = |a_2a_4 - \alpha a_3^2| = \left| \frac{c_1c_3}{24} + \frac{(9-8\alpha)c_1^2c_2}{144} + \frac{(3-4\alpha)c_1^4}{144} - \frac{\alpha c_2^2}{36} \right| \tag{3.4}$$

Suppose $\alpha \leq \frac{3}{4}$. Then

$$|H_2^{\alpha}(2)| \leq \frac{|c_1||c_3|}{24} + \frac{(3-4\alpha)}{144}|c_1|^4 + \frac{\alpha}{36}|c_2| \left| c_2 - \frac{(9-8\alpha)}{2\alpha}\frac{c_1^2}{2} \right|$$

which by lemmas (2) and (3) noting that $\frac{9-8\alpha}{2\alpha} \geq 2$ for $\alpha \leq \frac{3}{4}$ gives

$$|H_2^{\alpha}(2)| \leq 1 - \alpha.$$

Substituting for c_2 and c_3 in (3.4) using lemma (5), we obtain

$$|H_2^\alpha(2)| = \left| \frac{(1-\alpha)}{16}c_1^4 + \frac{(5-4\alpha)c_1^2(4-c_1^2)x}{96} \right. \\ \left. - \frac{\left[3c_1^2+2\alpha(4-c_1^2)\right](4-c_1^2)x^2}{288} + \frac{c_1(1-|x|^2)(4-c_1^2)z}{48} \right| \tag{3.5}$$

We write (3.5) as

$$|H_2^\alpha(2)| = \left| \frac{(\alpha-1)}{16}c_1^4 - \frac{(5-4\alpha)c_1^2(4-c_1^2)x}{96} \right. \\ \left. + \frac{\left[3c_1^2+2\alpha(4-c_1^2)\right](4-c_1^2)x^2}{288} - \frac{c_1(1-|x|^2)(4-c_1^2)z}{48} \right|$$

Letting $c_1 = c$, assuming without restriction that $c \in [0,2]$ and $\rho = |x|$, we obtain

$$\begin{aligned} |H_2^\alpha(2)| &\le \frac{(\alpha-1)c^4}{16} + \frac{c(4-c^2)}{48} + \frac{(5-4\alpha)c^2(4-c^2)\rho}{96} \\ &+ \frac{(4-c^2)(c-2)[(3-2\alpha)c-4\alpha]\rho^2}{288} \\ &= F(\rho) \end{aligned}$$

Then,

$$F'(\rho) = \frac{(5-4\alpha)c^2(4-c^2)}{96} + \frac{(4-c^2)(c-2)[(3-2\alpha)c-4\alpha]\rho}{144}$$

which is positive whenever $\frac{3}{4} < \alpha \le \frac{5}{4}$. Hence $F(\rho)$ is increasing on $[0,1]$ so that $F(\rho) \le F(1)$. Thus we have

$$|H_2^\alpha(2)| \le F(1) \le \frac{(8\alpha-9)c^4}{72} + \frac{(9-8\alpha)c^2}{36} + \frac{\alpha}{9} = G(c)$$

The maximum of $G(c)$ on $[0,2]$ occurs at $c=1$ if $\frac{3}{4} < \alpha \le \frac{9}{8}$ while it is at $c=2$ for $\frac{9}{8} \le \alpha \le \frac{5}{4}$ and is thus given by $G(1) = \frac{8\alpha-9}{72} + \frac{9-8\alpha}{36} + \frac{\alpha}{9} = \frac{1}{8}$ and $G(2) = \alpha - 1$ suppose $\alpha > \frac{5}{4}$. Then we write (3.5) as

$$|H_2^\alpha(2)| = \left| \frac{(\alpha-1)c_1^4}{16} + \frac{(4\alpha-5)c_1^2(4-c_1^2)x}{96} \right. \\ \left. + \frac{\left[3c_1^2+2\alpha(4-c_1^2)\right](4-c_1^2)x^2}{288} - \frac{c_1(1-|x|^2)(4-c_1^2)z}{48} \right|$$

By similar argument

$$\begin{aligned}|H_2^\alpha(2)| &\le \frac{(\alpha-1)c^4}{16} + \frac{c(4-c^2)}{48} + \frac{(4\alpha-5)c^2(4-c^2)\rho}{96}\\ &+ \frac{(4-c^2)(c-2)\left[(3-2\alpha)c-4\alpha\right]\rho^2}{288}\\ &= F(\rho)\end{aligned}$$

$$F'(\rho) = \frac{(4\alpha-5)c^2(4-c^2)}{96} + \frac{(4-c^2)(c-2)\left[(3-2\alpha)c-4\alpha\right]\rho}{144}$$

which is negative whenever $\alpha > \frac{5}{4}$. Hence $F(\rho)$ is decreasing on $[0,1]$, so that $F(\rho) \le F(0)$. Therefore,

$$\begin{aligned}|H_2^\alpha(2| &\le \frac{(\alpha-1)c^4}{16} + \frac{c(4-c^2)}{48}\\ &= G(c)\end{aligned}$$

we find that $G(c) \le G(2) = \alpha - 1$ which yields for $\alpha \ge \frac{9}{8}$

$$|H_2^\alpha(2)| \le \alpha - 1$$

we thus conclude that

$$|H_2^\alpha(2)| \le \begin{cases} 1-\alpha, & \text{if } 0 \le \alpha \le \frac{3}{4},\\ \frac{1}{8}, & \text{if } \frac{3}{4} < \alpha \le \frac{9}{8},\\ \alpha - 1, & \text{if } \alpha \ge \frac{9}{8}.\end{cases}$$

Corollary 9. *[7] Let $f \in C$. Then*

$$|H_2(2)| \le \frac{1}{8}$$

Theorem 10. *Let $f \in C$. Then for real numbers $\lambda_j, j = 1, 2, 3, ...$*

$$\begin{aligned}|H_3^{\lambda_1,\lambda_2,\lambda_3}(1)| &\le \max\left\{\frac{\lambda_1}{3}, |\lambda_2-\lambda_1|\right\} + \max\left\{\frac{\lambda_2}{8}, |\lambda_3-\lambda_2|\right\}\\ &+ \max\left\{\frac{\lambda_1}{6}, |\lambda_1-\lambda_3|\right\}\end{aligned}$$

Corollary 11. *Let $f \in C$. Then*

$$|H_3^{1,1,1}(1)| \le \frac{15}{24}, |H_3^{1,1,2}(1)| \le \frac{7}{3}, |H_3^{1,2,1}(1)| \le \frac{13}{6}, |H_3^{2,1,1}(1)| \le \frac{17}{8}$$

$$|H_3^{1,2,2}(1)| \le \frac{9}{4}, |H_3^{2,1,2}(1)| \le \frac{7}{3}, |H_3^{2,2,1}(1)| \le \frac{8}{3}, |H_3^{1,3,2}(1)| \le 4$$

$|H_3^{1,1,1}(1)| = |H_3(1)| \le \frac{15}{24}$ was reported in Babalola [2].

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