Acta Universitatis Apulensis
ISSN: 1582-5329
http://www.uab.ro/auajournal/

No. 47/2016
pp. 89-97
doi: 10.17114/j.aua.2016.47.07

THIRD HANKEL DETERMINANT FOR CERTAIN SUBCLASS OF $P-$VALENT FUNCTIONS WHOSE RECIPROCAL DERIVATIVE HAS A POSITIVE REAL PART

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Abstract. The objective of this paper is to obtain an upper bound to the $H_3(p)$ Hankel determinant for certain subclass of p-valent functions, using Toeplitz determinants.

2010 *Mathematics Subject Classification*: 30C45; 30C50.

Keywords: keywords, phrases. $p-$valent function, upper bound, function whose reciprocal derivative has a positive real part, Hankel determinant, Toeplitz determinants.

1. Introduction

Let A_p (p is a fixed integer ≥ 1) denote the class of functions f of the form

$$f(z) = z^p + a_{p+1}z^{p+1} + \cdots \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$ with $p \in N = \{1, 2, 3, ...\}$. Let S be the subclass of $A_1 = A$, consisting of univalent functions. The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [12] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

This determinant has been considered by several authors in the literature. One can easily observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in S$. Further sharp bounds for the functional $|a_2a_4 - a_3^2|$ represents the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant (functional), given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2, \tag{1.3}$$

were obtained, for various subclasses of univalent and multivalent functions. Noonan et.al [9] was studied determined growth rate of second Hankel determinant of an a really mean $p-$valent function. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 3$ and $n = p$, denoted by $H_3(p)$, given by

$$H_3(p) = \begin{vmatrix} a_p & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix}. \tag{1.4}$$

For $f \in A_p,\ a_p = 1$, so that, we have

$$H_3(p) = a_{p+2}\big(a_{p+1}a_{p+3} - a_{p+2}^2\big) - a_{p+3}\big(a_{p+3} - a_{p+1}a_{p+2}\big) + a_{p+4}\big(a_{p+2} - a_{p+1}^2\big)$$

and by applying triangle inequality, we obtain

$$|H_3(p)| \leq |a_{p+2}||a_{p+1}a_{p+3} - a_{p+2}^2| + |a_{p+3}||a_{p+1}a_{p+2} - a_{p+3}| + |a_{p+4}||a_{p+2} - a_{p+1}^2|. \tag{1.5}$$

The sharp upper bound to the second Hankel functional, $H_2(2)$, for the subclass RT of S consisting of functions whose derivative has a real part, studied by Mac Gregor [8] was obtained by Janteng et al. [6]. For $f \in RT_p$, the sharp upper bound to $H_3(p)$ was obtained by Vamshee Krishna et al. [14]. For $f \in \widetilde{RT}p$, $|\ a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{2p}{(p+2)}\right]^2$ was obtained by Venkateswarlu et al. [16]. DVK et al. [15] was obtained $|\ a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{2p}{(p+2)}\right]^2$ for $f \in \widetilde{RT}p$.

Motivated by the result obtained by Babalola [2] in finding the sharp upper bound to the Hankel determinant in this paper, we obtain an upper bound to the functional $|a_{p+1}a_{p+2} - a_{p+3}|$ and hence for $|H_3(p)|$, for the function f given in (1.1), belonging to the class $\widetilde{RT}_p$, defined as follows.

Definition 1. A function $f \in A_p$ is said to be function whose reciprocal derivative has a positive real part (also called reciprocal of bounded turning functions), denoted by $f \in \widetilde{RT}_p$, if and only if

$$Re\left[\frac{pz^{p-1}}{f'(z)}\right] > 0, \quad \forall \ \ z \in E. \tag{1.6}$$

For choice if $p = 1$, we obtain $\widetilde{RT}_1 = \widetilde{RT}$. Some preliminary lemmas required for proving our result are as follows:

2. Preliminary Results

Let $\mathscr{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = \left[1 + \sum_{n=1}^{\infty} c_n z^n\right], \tag{2.1}$$

which are regular in the open unit disc E and satisfy $\text{Re}\{p(z)\} > 0$ for any $z \in E$. Here $p(z)$ is called the Caratheòdory function [3].

Lemma 1. [11, 13] *If $p \in \mathscr{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2. [5] *The power series for $p(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n$ given in (2.1) converges in the open unit disc E to a function in $\mathscr{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \ n = 1, 2, 3, \cdots$$

and $c_{-k} = \overline{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum\limits_{k=1}^{m} \rho_k P_0(e^{it_k} z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $P_0(z) = \left(\frac{1+z}{1-z}\right)$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in [5] is due to Caratheòdory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2, for $n = 2$, we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4|c_1|^2] \geq 0,$$

which is equivalent to

$$2c_2 = c_1^2 + x(4 - c_1^2), \text{ for some } x, \ |x| \leq 1. \tag{2.2}$$

For $n = 3$,

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} \geq \ 0.$$

and is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \quad (2.3)$$

From the relations (2.2) and (2.3), after simplifying, we get

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z,$$
$$\text{for some real value of } z, \text{ with } |z| \leq 1. \quad (2.4)$$

To obtain our result, we refer to the classical method initiated by Libera and Zlotkiewicz [7] and used by several authors in the literature.

3. Main Result

Theorem 1. *If* $f(z) \in \widetilde{RT_p}$ *then*

$$|\, a_{p+1}a_{p+2} - a_{p+3}| \leq \left[\frac{\sqrt{2}p(p^2 + 3p + 6)^{\frac{3}{2}}}{3\sqrt{3}(p+1)(p+2)(p+3)} \right].$$

Proof. For $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in \widetilde{RT}_p$, there exists an analytic function $p \in \mathscr{P}$ in the open unit disc E with $p(0) = 1$ and $\text{Re}\{p(z)\} > 0$ such that

$$\frac{pz^{p-1}}{f'(z)} = p(z) \;\Leftrightarrow\; pz^{p-1} = p(z)f'(z). \quad (3.1)$$

Using the series representations for $f'(z)$ and $p(z)$ in (3.1), we have

$$pz^{p-1} = \Big(1 + \sum_{n=1}^{\infty} c_n z^n\Big)\Big(pz^{p-1} + \sum_{n=p+1}^{\infty} na_n z^{n-1}\Big).$$

Upon simplification, we obtain

$$\begin{aligned} 0 &= \{c_1p + (p+1)a_{p+1}\}\, z^p + \{c_2p + c_1(p+1)a_{p+1} + (p+2)a_{p+2}\}\, z^{p+1} \\ &\quad + \{c_3p + c_2(p+1)a_{p+1} + c_1(p+2)a_{p+2} + (p+3)a_{p+3}\}\, z^{p+2} \\ &\quad + \{c_4p + c_3(p+1)a_{p+1} + c_2(p+2)a_{p+2} + c_1(p+3)a_{p+3} + (p+4)a_{p+4}\}\, z^{p+3} + \cdots . \end{aligned} \quad (3.2)$$

Equating the coefficients of like powers of z^p, z^{p+1}, z^{p+2} and z^{p+3} respectively in (3.2), we can now write

$$a_{p+1} = \frac{-pc_1}{(p+1)}; \quad a_{p+2} = \frac{p}{p+2}(c_1^2 - c_2); \quad a_{p+3} = \frac{-p}{p+3}(c_3 - 2c_1c_2 + c_1^3);$$
$$a_{p+4} = \frac{-p}{p+4}(3c_2c_1^2 - 2c_3c_1 - c_1^4 - c_2^2 + c_4). \tag{3.3}$$

Substituting the values of a_{p+1}, a_{p+2} and a_{p+3} from (3.3) in the functional $\mid a_{p+1}a_{p+2} - a_{p+3} \mid$ for the function $f \in \widetilde{RT_p}$, after simplifying, we get

$$\mid a_{p+1}a_{p+2} - a_{p+3}| = \frac{p}{(p+1)(p+2)(p+3)}\Big| 2c_1^3 - c_1c_2(p^2+3p+4) + c_3(p+1)(p+2)\Big|. \tag{3.4}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively from Lemma 2 on the right-hand side of (3.4), we have

$$\mid 2c_1^3 - c_1c_2(p^2+3p+4) + c_3(p+1)(p+2)| = \Big| 2c_1^3 - \frac{c_1(p^2+3p+4)}{2}\{c_1^2 + x(4-c_1^2)\}$$
$$+ \frac{(p+1)(p+2)}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\}\Big|.$$

Using the fact $\mid z \mid < 1$, after simplifying, we get

$$4| 2c_1^3 - c_1c_2(p^2+3p+4) + c_3(p+1)(p+2)| \le \Big|8c_1^3 - 2c_1\{c_1^2 + x(4-c_1^2)\}(p^2+3p+4)$$
$$+ (p+1)(p+2)\{c_1^3 + 2c_1(4-c_1^2)x + 2(4-c_1^2) - x^2(4-c_1^2)(c_1-2)\}\Big|. \tag{3.5}$$

Since $c_1 = c \in [0,2]$, using the result $(c_1 + a) \ge (c_1 - a)$, where $a \ge 0$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of (3.5), we have

$$4| 2c_1^3 - c_1c_2(p^2+3p+4) + c_3(p+1)(p+2)| \le \Big|c^3(p^2+3p-2)$$
$$+ 2(4-c^2)(p+1)(p+2) + 4c(4-c^2)\mu + (c-2)(4-c^2)\mu^2(p+1)(p+2)\Big|$$
$$= F(c,\mu) \ , \ \ 0 \le \mu = \mid x \mid \le 1 \ \text{ and } \ 0 \le c \le 2. \tag{3.6}$$

We next maximize the function $F(c,\mu)$ on the closed region $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ given in (3.6) partially with respect to μ, we obtain

$$\frac{\partial F}{\partial \mu} = 2\Big[\mu(c-2)(p+1)(p+2) + 2c\Big](4-c^2) > 0. \tag{3.7}$$

For $0 < \mu < 1$ and for fixed c with $0 < c < 2$, from (3.7), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ becomes an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed region $[0,2]\times[0,1]$. Moreover, for a fixed $c\in[0,2]$, we have

$$\max_{0\le\mu\le1} F(c,\mu)=F(c,1)=G(c).$$

Therefore, replacing μ by 1 in $F(c,\mu)$, upon simplification, we obtain

$$G(c) = -8c^3+4c(p^2+3p+6) \tag{3.8}$$
$$G'(c) = -24c^2+4(p^2+3p+6) \tag{3.9}$$
$$G''(c) = -48c. \tag{3.10}$$

For optimum value of $G(c)$, consider $G'(c)=0$. From (3.9), we get

$$c^2 = \frac{p^2+3p+6}{6}.$$

Using the obtained value of $c=\sqrt{\frac{p^2+3p+6}{6}}$ in (3.10), then

$$G''(c) = -48\sqrt{\frac{p^2+3p+6}{6}}<0, \text{ for } p\in N.$$

Therefore, by the second derivative test, $G(c)$ has maximum value at $c=\sqrt{\frac{p^2+3p+6}{6}}$. Substituting the value of c in the expression (3.8), upon simplification, we obtain the maximum value of $G(c)$ as

$$G_{max} = 16\left[\frac{p^2+3p+6}{6}\right]^{\frac{3}{2}}. \tag{3.11}$$

From the expressions (3.6) and (3.11), we obtain

$$|\,2c_1^3-c_1c_2(p^2+3p+4)+c_3(p+1)(p+2)|\le 4\left[\frac{p^2+3p+6}{6}\right]^{\frac{3}{2}}. \tag{3.12}$$

Simplifying the relations (3.4) and (3.12), we obtain

$$|\,a_{p+1}a_{p+2}-a_{p+3}| \le \left[\frac{\sqrt{2}p(p^2+3p+6)^{\frac{3}{2}}}{3\sqrt{3}(p+1)(p+2)(p+3)}\right]. \tag{3.13}$$

This completes the proof of our Theorem.

Remark 1. For the choice of $p = 1$, from (3.13), we obtain $|a_2a_3 - a_4| \leq \frac{1}{6}\left(\frac{5}{3}\right)^{\frac{3}{2}}$, obtained by Babalola [2] and Venkateswarlu et al. [16]. From this we conclude that, for $p = 1$, the sharp upper bound to the $|a_{p+1}a_{p+2} - a_{p+3}|$ of a function whose derivative has a positive real part for $p-$valent function and a function whose reciprocal derivative has a positive real part for $p-$valent function is the same.

The following theorem is a straight forward verification on applying the same procedure as described in Theorem 1 and the result is sharp for the values $c_1 = 0, c_2 = 2$ and $x = 1$.

Theorem 2. *If* $f \in \widetilde{RT}p$ *then* $|\ a_{p+2} - a_{p+1}^2| \leq \left[\frac{2p}{p+2}\right]$.

Using the fact that $|c_n|,\ n \in\ N = \{1, 2, 3, \cdots\}$, with the help of c_2 and c_3 values given in (2.2) and (2.4) respectively together with the values in (3.3), we obtain $|a_k| \leq \frac{2p}{k}$, where $k \in \{p+1,\ p+2,\ p+3,\ \cdots\}$.

Substituting the results of Theorems 1, 2, $|a_k| \leq \frac{2p}{k}$ where $k \in \{p+1,\ p+2,\ p+3, \cdots\}$ and $|\ a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{2p}{(p+2)}\right]^2$ in (1.5), we obtain the following corollary.

Corollary 3. *If* $f(z) \in \widetilde{RT_p}$ *then*

$$|\ H_3(p)| \leq \frac{2p^2}{p+2}\left[\frac{4p}{(p+2)^2} + \frac{\sqrt{2}(p^2+3p+6)^{\frac{3}{2}}}{3\sqrt{3}(p+1)(p+3)^2} + \frac{2}{p+4}\right]. \tag{3.14}$$

Remark 2. For the choice $p = 1$, from the expressions (3.14), we obtain $|H_3(1)| \leq 0.7422$. These inequalities are sharp and coincide with the results of Babalola [2] and Venkateswarlu et al. [16]. From this we conclude that, for $p = 1$, the sharp upper bound to the third Hankel determinant of a function whose derivative has a positive real part for $p-$valent function and a function whose reciprocal derivative has a positive real part for $p-$valent function is the same.

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